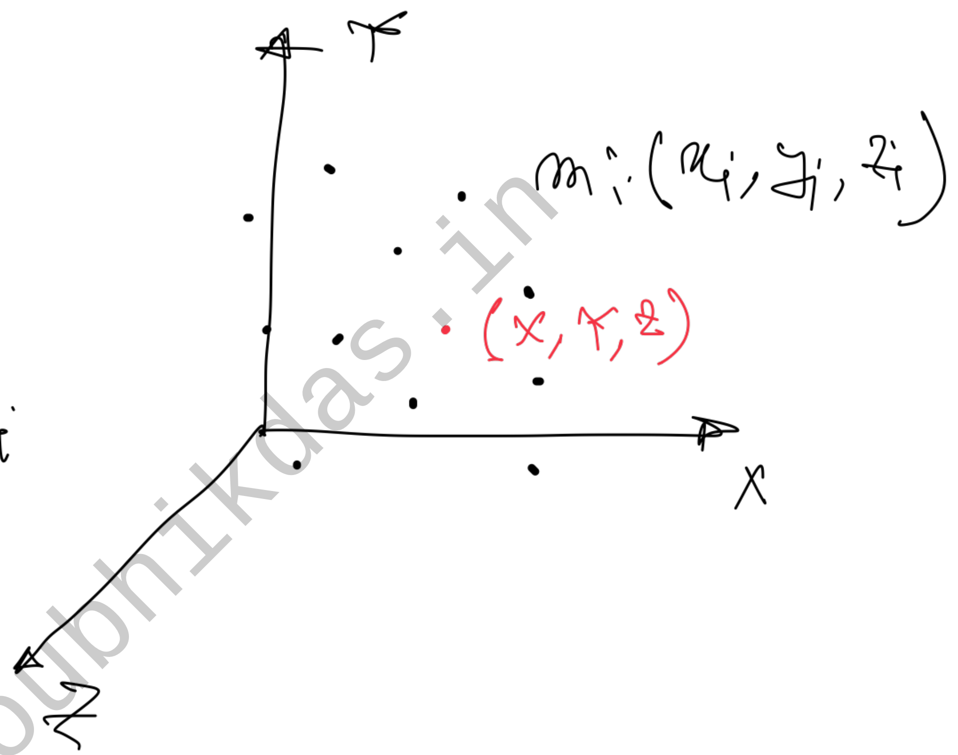
Centre of Mass, Linear Momentum, Collision

Examp- Spin Ball

- Let the mass of ith particle be m_i and its coordinates with reference to the chosen axes be x_i, y_i, z_i



$$X = \frac{1}{M}\sum_i m_i x_i$$

$$Y = \frac{1}{M}\sum_i m_i y_i$$

$$Z = \frac{1}{M}\sum_i m_i z_i$$

where,

$M = \sum_i m_i$ is the total mass of the system.

If the position vector of the ith particle is $\vec{r}_i$, the centre of mass is defined to have the position vector,

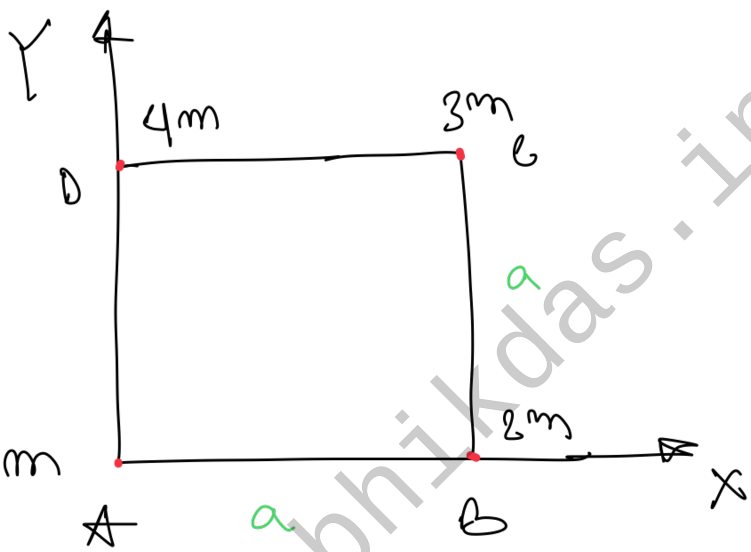
$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i$$

Taking x, y, z components of the eq^n, we

Get the coordinates of centre of mass.

$$\boxed{X = \frac{1}{M}\sum_i m_i x_i \; ; \; Y = \frac{1}{M}\sum_i m_i y_i \; ; \; Z = \frac{1}{M}\sum_i m_i z_i}$$

Example



Find the centre of mass of the particle system.

Particle	mass	X	Y
A	m	0	0
B	2m	a	0
C	3m	a	a
D	4m	0	a

The coordinates of the centre of mass,

$$X = \frac{m.0 + 2m.a + 3m.a + 4m.0}{m + 2m + 3m + 4m}$$

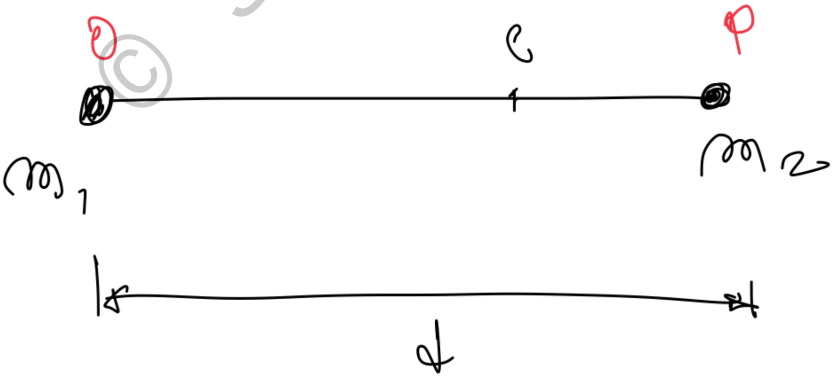
$$= \frac{a}{2}$$

$$Y = \frac{m.0 + 2m.0 + 3m.a + 4m.a}{m + 2m + 3m + 4m}$$

$$= \frac{7a}{10}$$

The centre of mass is ~~at~~, $\left(\frac{a}{2}, \frac{7a}{20}\right)$

Centre of mass of two particles



Let $m_1 (0, 0, 0)$

$m_2 (d, 0, 0)$

The total mass,

$M = m_1 + m_2$

$$\sum_i m_i x_i = m_1 \cdot 0 + m_2 \cdot d = m_2 d$$

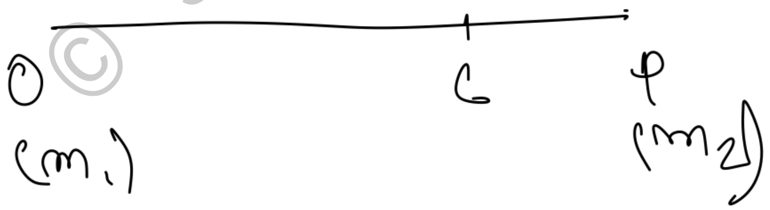
$$\sum_i m_i y_i = 0 \; ; \; \sum_i m_i z_i = 0$$

So, by definition,

$$X = \frac{1}{M} \sum_i m_i x_i$$

$$= \frac{1}{M} \cdot m_2 d = \frac{m_2 d}{m_1 + m_2}$$

The centre of mass is at $\left(\frac{m_2 d}{m_1 + m_2}, 0, 0 \right)$



If O, C, P be the positions of m_1, the centre of mass and m_2 respectively, we have

$$OC = \frac{m_2 d}{m_1 + m_2} \text{ and } CP = \frac{m_1 d}{m_1 + m_2}$$

So that,

$$m_1(OC) = m_2(CP)$$

The centre of mass divides internally the line joining the two particles in inverse ratio of their masses.

$$X = \frac{M_1X_1 + M_2X_2}{M_1+M_2} \; ; \; Y = \frac{M_1Y_1 + M_2Y_2}{M_1+M_2}$$

$$Z = \frac{M_1Z_1 + M_2Z_2}{M_1+M_2}$$

If we know the centres of mass of parts of the system and their masses, we can get the combined centre of mass by treating the parts as point particles placed at their respective centres of mass.

Centre of mass of Continuous Bodies

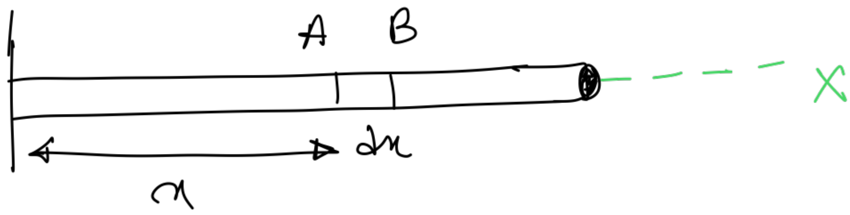
- If we consider the body to have continuous distribution of matter, the summation in the formula should be replaced by integration.

We do not talk of the i^{th} particle, rather of a small element of the body having a mass dm.

If x, y, z are the coordinates of the small mass dm, the centre of mass is at,

$$X = \frac{1}{M}\int x\,dm, \quad Y = \frac{1}{M}\int y\,dm; \quad Z = \frac{1}{M}\int z\,dm.$$

(a) Uniform Straight Rod



M = Mass ; L = length.

So, Mass per unit of length = M/L

Consider the block between x & $(x+dx)$

Hence, the mass of the element,

$$dm = \left(\frac{M}{L}\right) dx$$

So, the coordinates of the centre of mass is,

$$X = \frac{1}{M}\int_L x \cdot dm$$

$$= \frac{1}{M}\int_0^L x \cdot \left(\frac{M}{L}\right) dx$$

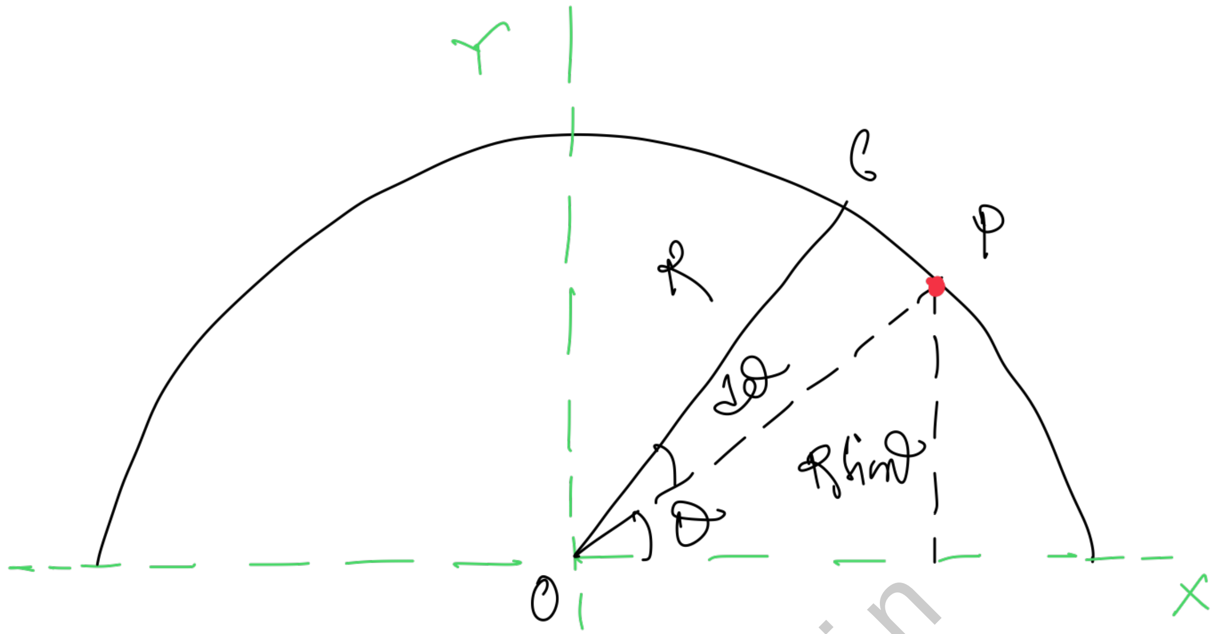
$$= \frac{1}{L}\left[\frac{1}{2}x^2\right]_0^L = \frac{L}{2}.$$

Y and Z coordinates are 0.

So, the coordinate of centre of Mass

$\left(\frac{L}{2}, 0, 0\right)$. i.e at the middle point of the rod.

(b) Uniform Semicircular Wire



Let M = Mass

R = Radius

So, Mass per unit length $= \frac{M}{\pi R}$

The coordinates of the element $(R\cos\theta, R\sin\theta)$

Mass of CP i.e. $R\,d\theta$ element,

$$dm = \left(\frac{M}{\pi R}\right) R\,d\theta = \frac{M}{\pi} d\theta$$

The coordinates of the centre of Mass –

$$X = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^{\pi} (R\cos\theta) \cdot \frac{M}{\pi} \cdot d\theta$$

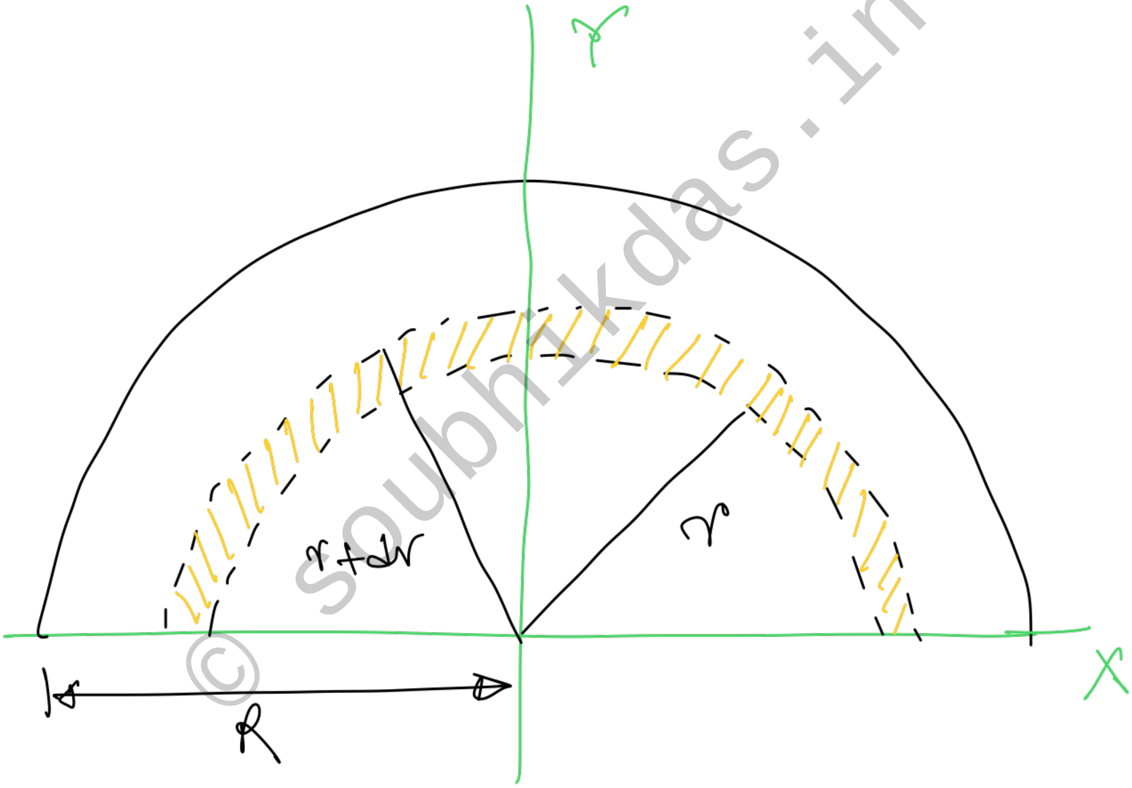
$$= 0$$

$$Y = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\theta)\cdot\frac{M}{\pi}\,d\theta$$

$$= \frac{2R}{\pi}.$$

The Center of Mass is at $\left(0, \frac{2R}{\pi}\right)$.

⑥ $\boxed{\text{Uniform Semicircular Plate}}$



Let M = Mass ; R = Radius

Unit mass per area $= \frac{M}{\pi R^2/2} = \frac{2M}{\pi R^2}$

The area of the shaded part $= \pi r\,.dr$

Hence the Mass of the semicircular element,

$$\frac{2M}{\pi R^2}\left(\pi r.dr\right) = \frac{2Mr\,dr}{R^2}$$

The Y. coordinate of the centre of mass of this wire (element) is $2r/\pi$ [from ⓑ]

So, the Y. coordinate of centre of mass of the plate,

$$Y = \frac{1}{M}\int_0^R \frac{2r}{\pi}\cdot\frac{2Mr\,dr}{R^2}$$

$$= \frac{4}{\pi R^2}\int_0^R r^2\,dr$$

$$= \frac{4}{\pi R^2}\left[\frac{r^3}{3}\right]_0^R$$

$$= \frac{4R}{3\pi}$$

& X. coordinate is zero. by symmetry.

$$= \left(0, \frac{4R}{3\pi}\right)$$

- When external forces do not add up to zero, the centre of mass is accelerated and the acceleration is given as -

$$\vec{a}_{cm} = \frac{\vec{F}_{ext}}{M}$$

Then, the motion of the centre of mass of a system is identical to the motion of a single particle of mass to the mass of the given system, acted upon by the same external forces that act on the system.

Linear Momentum and Its Conservation Principle

- The (Linear) momentum of a particle is defined as $\vec{p} = m\vec{v}$

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$$

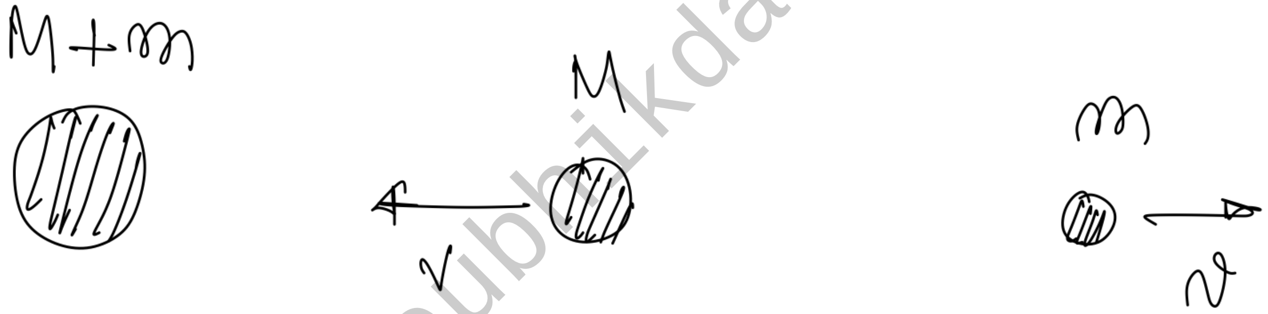
If the external forces acting on the system add up to zero, the centre of mass moves with constant velocity.

i.e. $\vec{P}$ = constant.

Linear momentum of a system remains constant (in magnitude and direction) if the external forces acting on the system add up to zero. This is known as the principle of conservation of linear momentum.

- Radioactivity (By Internal force)

Nucleus $\Rightarrow$ Residual Nucleus + alpha particle.



After the emission, the system is broken up into two parts. If the alpha particle is emitted with a speed v, the residual nucleus must recoil in the opposite direction with a speed V, so that

$$M\vec{V} + m\vec{v} = 0 \quad \text{or,} \quad \vec{V} = -\frac{m}{M}\vec{v}$$

Rocket Propulsion

At $t = 0$, let the mass of a rocket together with its fuel is M_0.

Let the gas is ejected at a constant rate $r = -\frac{dM}{dt}$.

The gas is ejected at a constant velocity u with respect to the rocket.

At time t, the mass of the rocket together with the remaining fuel,

$$M = M_0 - rt$$

If the velocity of the rocket at time t is v, the linear momentum of this mass M is,

$$P = Mv = (M_0 - rt)v \quad \text{——— (1)}$$

Consider a small time interval Δt, a mass $\Delta M = r\Delta t$ of the gas is ejected. The velocity of the rocket becomes $v + \Delta v$.

The velocity of the gas with respect to ground is,

$$\vec{v}_{gas,\,ground} = \vec{v}_{gas,\,rocket} + \vec{v}_{rocket,\,ground}$$

$$= -u + v \quad [\text{in forward direction}]$$

The linear momentum of the mass M at $t+\delta t$ is,

$$(M - \Delta M)(v + \Delta v) + \Delta M (v - u) \text{ —— } (2)$$

Assuming no external force on the rocket - fuel system, from (1) & (2)

$$(M - \Delta M)(v + \Delta v) + \Delta M (v - u) = Mv$$

or, $$\Delta v = \frac{\Delta M \cdot u}{M - \Delta M}$$

or, $$\frac{\Delta v}{\Delta t} = \frac{\Delta M}{\Delta t} \cdot \frac{u}{M - \Delta M} = \frac{r u}{M - r\Delta t}$$

for $\Delta t \to 0$

$$\frac{dv}{dt} = \frac{r u}{M} = \frac{r u}{M_0 - rt} \text{ —— } (3)$$

Neglecting any external force such as gravity, from eq^n (3) —

$$\int_0^v dv = \int_0^t \frac{r u \, dt}{M_0 - rt}$$

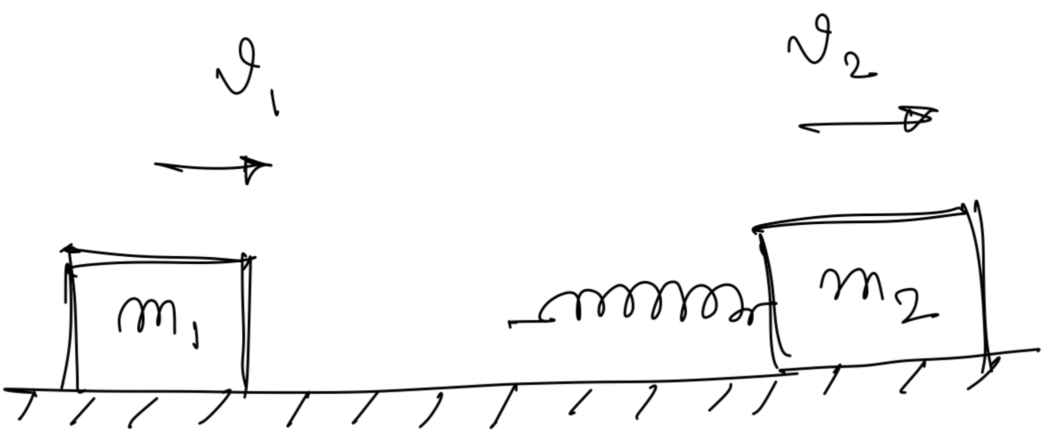
or, $$v = r u \int_0^t \frac{dt}{M_0 - rt}$$

$$= r u \left(-\frac{1}{r}\right) \ln \frac{M_0 - rt}{M_0}$$

or, $$\boxed{v = u \ln \frac{M_0}{M_0 - rt}}$$

Collision

$v_1 > v_2$



If after collision the speed of m_1 & m_2 are v_1' & v_2' respectively, then by conservation of momentum, $v_2' > v_1'$

$$\boxed{m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'}$$

~~And~~ by conservation of energy,

$$\boxed{\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2}$$

- The spring will be compressed until the speed of the blocks become same, and at that point the energy will be,

$$\boxed{\frac{1}{2} m_1 V^2 + \frac{1}{2} m_2 V^2 + \frac{1}{2} K x_{max}^2 = E}$$

K – spring constant

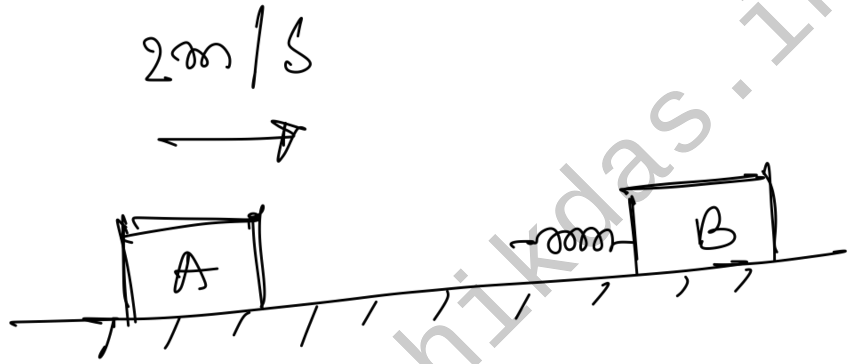
x_{max} – length of the compressed spring

V – speed of m_1 & m_2.

The kinetic energy before the collision is same as the kinetic energy after the collision.

- The speed of m_1 will decrease and m_2 will increase.

Example



Mass of A & B are 1kg each.

If $K = 50\ N/m$, find the max^m compression of the spring.

- Max^m compression will take place when the blocks move with equal velocity.

If V is the common speed at max^m compression,

$$m \cdot 2 + m \cdot 0 = m \cdot v + m \cdot v$$

$$m = 1\ kg.$$

or, $v = 1\ m/s$

Initial kinetic energy,

$$\frac{1}{2} \cdot (1 kg) \cdot (2 m/s)^2 = 2 \text{ Joule}$$

Final kinetic energy,

$$\frac{1}{2} (1 kg) (1 m/s)^2 + \frac{1}{2} (1 kg) (1 m/s)^2$$

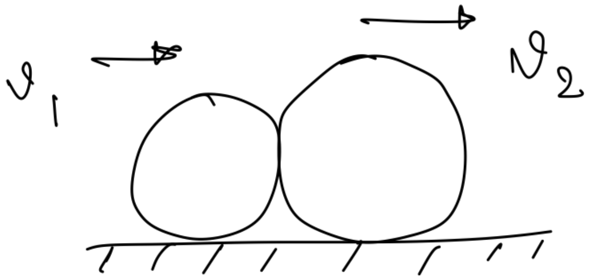
$$= 1 \text{ Joule}$$

The kinetic energy lost is stored as the elastic energy in the spring.

Hence,

$$\frac{1}{2} (50 N/m) \cdot x_{max}^2 = 2J - 1J = 1J$$

or, $x = 0.2\ m$



If the initial kinetic energy is equal to the final kinetic energy, the collision is called an elastic collision. Also if the balls remains deformed, (depends on the material) is inelastic collision.

Thus for elastic collision,

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

i.e. $K_f = K_i$

for Inelastic collision, $v_1' = v_2' = V$

$$m_1 v_1 + m_2 v_2 = m_1 V + m_2 V$$

and $K_f < K_i$

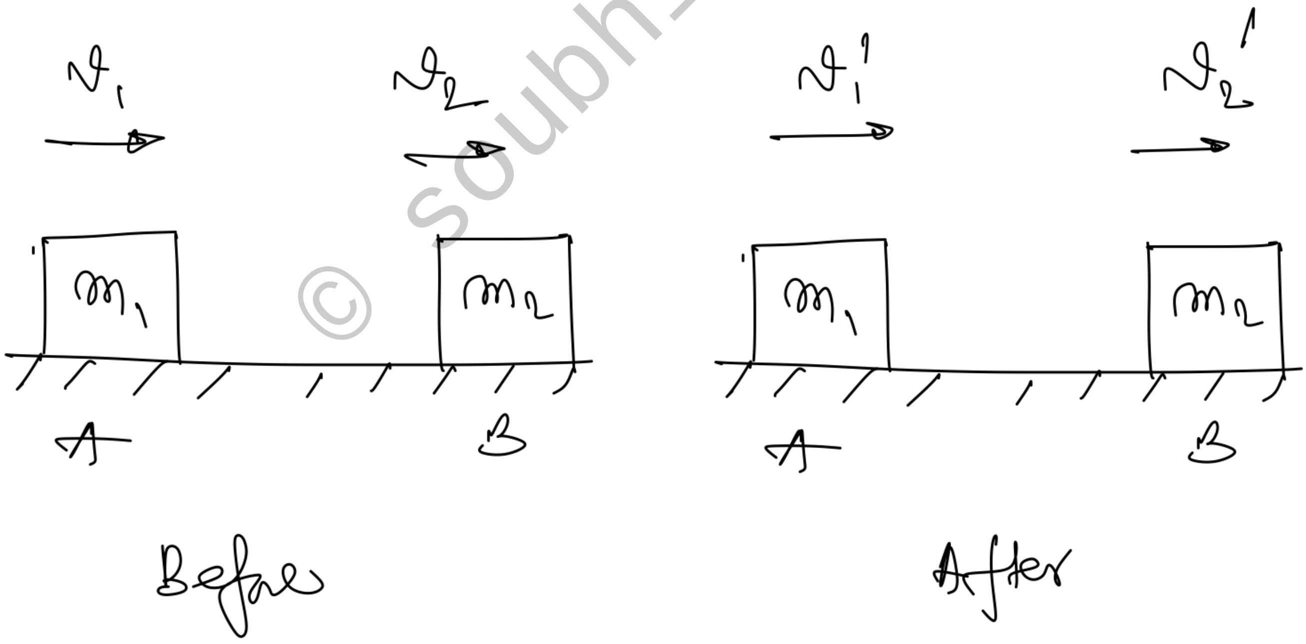
For partially elastic collision,

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$K_f < K_i$; $\Delta K = K_f - K_i$.

— ΔK is less for partially elastic collision than perfectly inelastic collision.

III Elastic collision in one dimension



Assume, $v_1 > v_2$

Total linear momentum remains constant

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

or, $m_1(v_1 - v_1') = m_2(v_2' - v_2)$

The kinetic energy before and after collision are equal:-

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

or, $m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$

we will get,

$$v_1 - v_2 = v_2' - v_1'$$

i.e. velocity of seperation (after collision) = velocity of approach (before collision)

$$\boxed{\begin{aligned} v_1' &= \frac{(m_1 - m_2)}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \\ v_2' &= \frac{2m_1}{m_1 + m_2} v_1 - \frac{(m_1 - m_2)}{m_1 + m_2} v_2 \end{aligned}}$$

Special case

① Elastic collision between a heavy body and a light body.

- let $m_1 \gg m_2$

$$\frac{m_1 - m_2}{m_1 + m_2} \approx 1 ; \quad \frac{2m_2}{m_1 + m_2} \approx 0$$

and $\frac{2m_1}{m_1 + m_2} \approx 2$

So, $v_1' \approx v_1$ and $v_2' \approx 2v_1 - v_2$

- let $m_2 \gg m_1$ i.e a light body hits a heavy body from behind

$$\frac{m_1 - m_2}{m_1 + m_2} \approx -1 ; \quad \frac{2m_2}{m_1 + m_2} \approx 2$$

and $\frac{2m_1}{m_1 + m_2} \approx 0$

Then $v_1' \approx 2v_2 - v_1$; $v_2' \approx v_2$

② Elastic collision of two bodies of equal mass.

- If $m_1 = m_2$

$$v_1' = v_2 \quad \text{and} \quad v_2' = v_1$$

Their velocities are mutually interchanged.

III Perfectly Inelastic collision in one dimension.

- Initial velocities for m_1 & m_2 are v_1 & v_2 respectively. If final velocity is V,

$$m_1 v_1 + m_2 v_2 = m_1 V + m_2 V$$

or,
$$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Loss in kinetic Energy —

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}(m_1+m_2)v^2$$

$$= \frac{m_1m_2(v_1-v_2)^2}{2(m_1+m_2)}$$

We see this loss in kinetic energy is positive.

Coefficient of Restitution

In general, the bodies are neither perfectly elastic nor perfectly inelastic.
In that case

velocity of separation = e (velocity of approach)

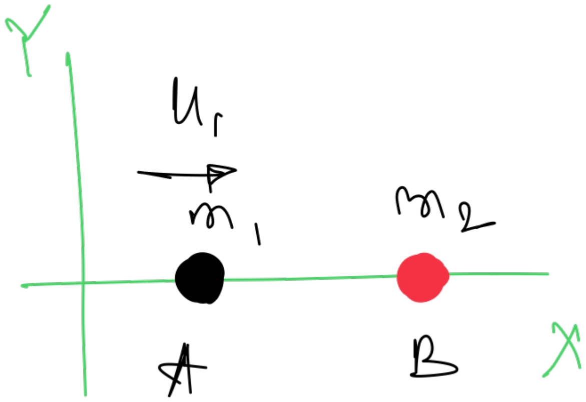
where $0 < e < 1$

e depends on the material and called coefficient of Restitution.

If $e = 1$. Perfectly elastic collision

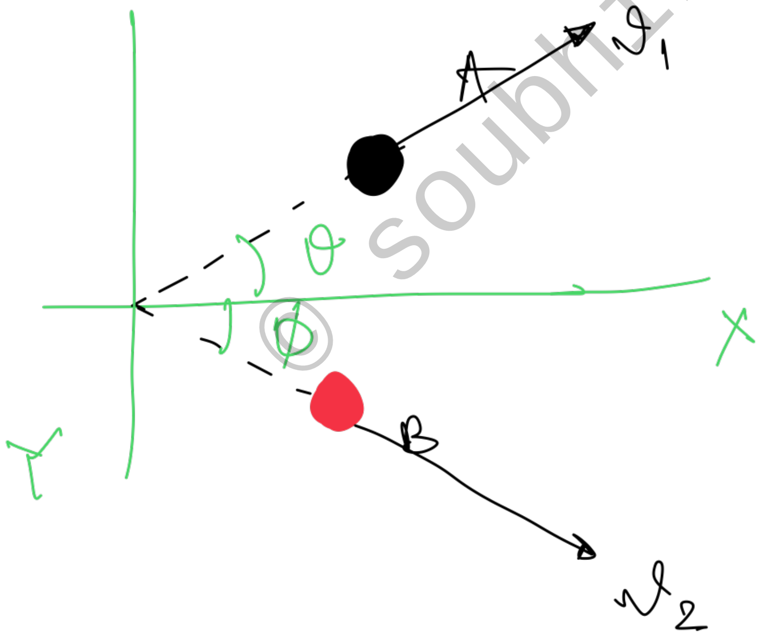
$e = 0$ Perfectly Inelastic collision.

III Elastic collision in Two Direction.



B is at rest and A moves towards B with a speed u_1.

— If the collision is not head on (the force during the collision is not along the initial velocity), the objects move along different lines with different velocity.



$$m_1 u_1 = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$$

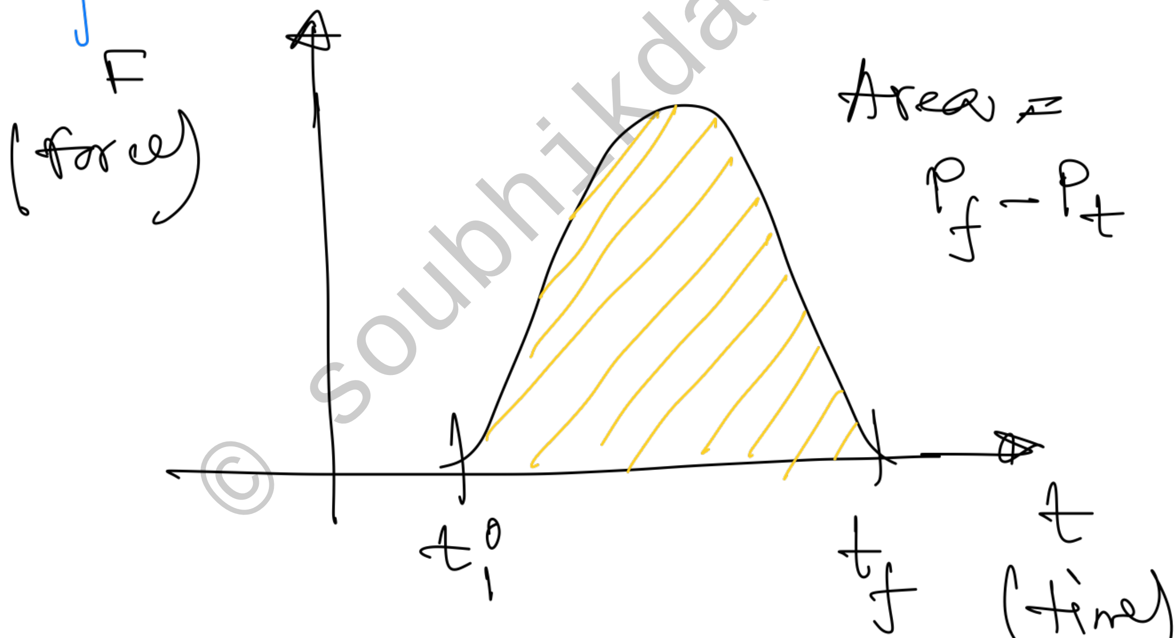
and. $$0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\phi$$

and, $K_f = K_i$

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

– The momentum of each object must be individually conserved in the direction perpendicular to the force.

Impulse & Impulsive Force



change in momentum produced by such Impulsive force is,

$$\vec{P}_f - \vec{P}_i = \int_{P_i}^{P_f} d\vec{P} = \int_{t_i}^{t_f} \frac{d\vec{P}}{dt} \cdot dt = \int_{t_i}^{t_f} \vec{F} \cdot dt.$$