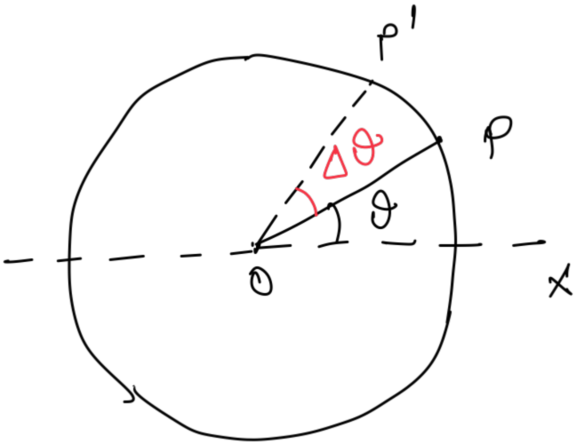
Circular Motion



- P is a particle moving in a circle of radius r.
- θ the angular position at a given instant.

In time Δt, θ increases to $\theta + \Delta\theta$.

- The rate of change of angular position is called angular velocity (ω).

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

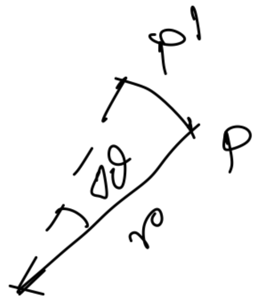
Angular acceleration,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\boxed{\begin{aligned} \theta &= \omega_0 t + \tfrac{1}{2}\alpha t^2 \\ \omega &= \omega_0 + \alpha t \\ \omega^2 &= \omega_0^2 + 2\alpha\theta \end{aligned}}$$

- Where ω_0 and ω are the angular velocities at time $t = 0$ & at time t.

- The linear distance PP' travelled by the particle in time Δt is,



$$\Delta S = r\,\Delta\theta$$

$$\frac{\Delta S}{\Delta t} = r\,.\,\frac{\Delta\theta}{\Delta t}$$

or, $\boxed{v = r\omega}$

- where v is the linear speed of the particle.

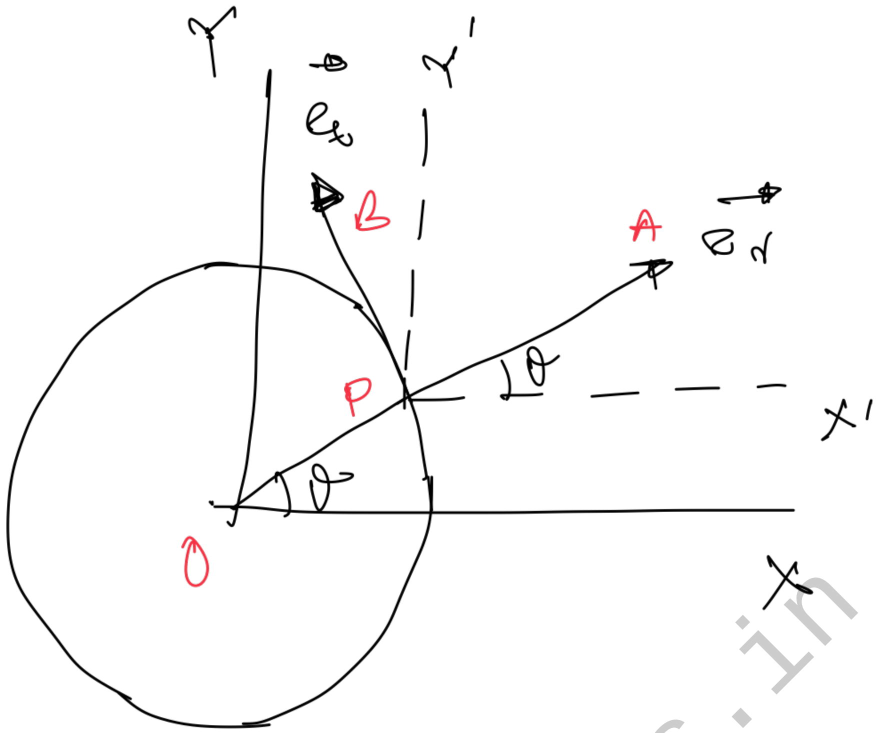
Now the rate of change of speed (not velocity)

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$$

Angular Acceleration

- a_t is not equal to net acceleration. a_t is the component of acceleration along the tangent. It is called the tangential acceleration.

VIII Unit vectors along the radius and the tangent



— $\vec{e_r}$ along the outward radius — radial unit vector.

— $\vec{e_\theta}$ in the direction of increasing θ, tangential unit vector.

$$\vec{PA} = \hat{i}\, PA\cos\theta + \hat{j}\, PA\sin\theta$$

$$\text{or, } \vec{e_r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

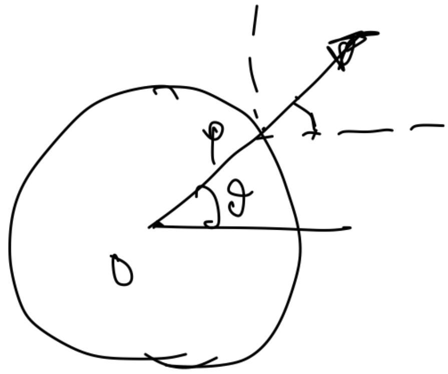
where $\hat{i}$ & $\hat{j}$ are the unit vectors along the x axis.

Similarly,

$$\vec{PB} = \hat{j}\, PB\cos\theta - \hat{i}\, PB\sin\theta$$

$$\text{or, } \vec{e_\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

Acceleration in circular motion



Now, $\vec{r} = \vec{OP}$

$$= OP\,\vec{e_r}$$

$$= r\left(\vec{i}\cos\theta + \vec{j}\sin\theta\right)$$

Now, velocity of the particle at time t,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[r\left(\vec{i}\cos\theta + \vec{j}\sin\theta\right)\right]$$

$$= r\left[\vec{i}\left(-\sin\theta\,\frac{d\theta}{dt}\right) + \vec{j}\left(\cos\theta\,\frac{d\theta}{dt}\right)\right]$$

$$= r\omega\left[-\vec{i}\sin\theta + \vec{j}\cos\theta\right]$$

$$= r\omega\,\vec{e_\theta}$$

Thus the velocity of the particle at any instant is along the tangent to the circle and its magnitude is

$$v = r\omega$$

- Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = -\omega^2 r\,\vec{e_r} + \frac{dv}{dt}\,\vec{e_\theta}$

III Uniform circular motion

- If the particle moves in the circle with a uniform speed, we call it a uniform circular motion.

$$\frac{d\omega}{dt} = 0 \quad \text{So, } \vec{a} = -\omega^2 r \vec{e_r}$$

Thus, acceleration of the particle is in the direction of $-\vec{e_r}$, i.e., towards the centre. The magnitude of acceleration,

$$a_r = \omega^2 r$$

$$= \frac{v^2}{r^2} \cdot r = \frac{v^2}{r}$$

- This acceleration is called centripetal acceleration.

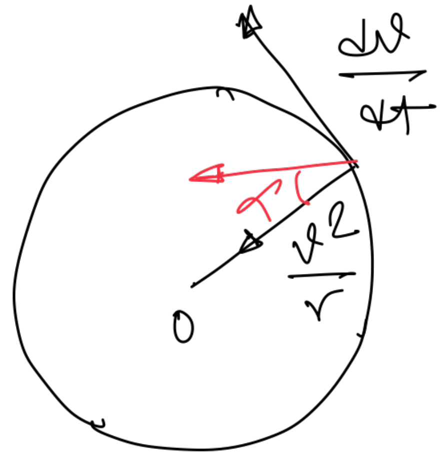
III Non Uniform circular motion

$$r = -\omega^2 r = -v^2/r$$

$$t = \frac{dv}{dt}$$

$$= \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

- If the direction of this resultant acceleration makes an angle α with the radius, where



$$\tan\alpha = \left(\frac{dv}{dt}\right) \Big/ \, v^2/r$$

Dynamics of circular motion

- If a particle moves in a circle as seen from an inertial frame, a resultant non-zero force must act on the particle.

- If the speed is constant, the acceleration of the particle is towards the centre and its magnitude is v^2/r.

The resultant force must act towards the centre and its magnitude F must satisfy -

$$a = F/m$$

or, $v^2/r = F/m$ or, $\boxed{F = \frac{mv^2}{r}}$

– The Force is called <u>centripetal Force</u>.

– A centripetal force of magnitude mv^2/r is needed to keep the particle in uniform circular motion.

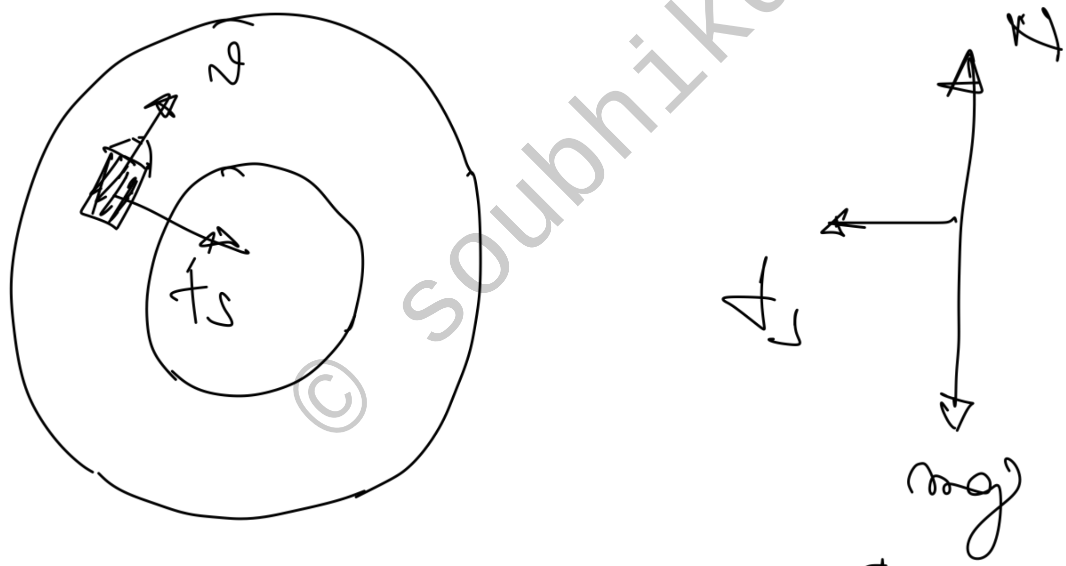
III) Circular Turning and Banking of Roads.

– External forces acting on the vehicle

(i) weight mg

(ii) Normal contact force N

(iii) Friction, f_s



The vehicle is not skidding –

centripetal force = Static friction

$$\frac{mu^2}{r} = f_s$$

Now, $f_s = \mu_s N = \mu_s mg$

So, $\dfrac{mv^2}{r} = \mu_s mg$

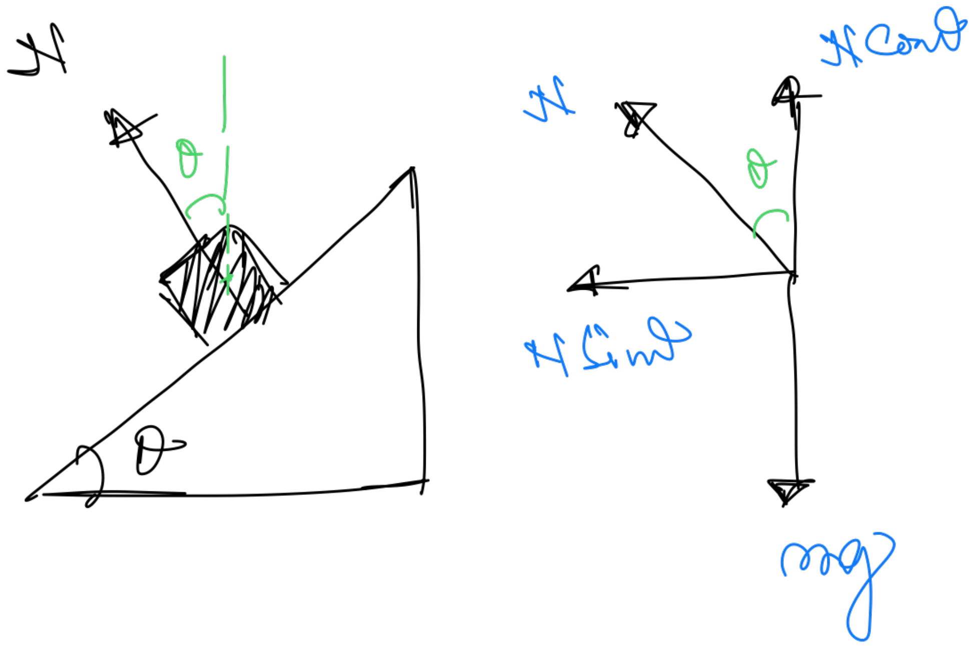
$$\mu_s = \frac{v^2}{rg}$$

- The magnitude of friction f_s cannot exceed $\mu_s N$. For vertical equilibrium $N = mg$, so that,

$$f_s \leq \mu_s mg$$

or, $$\boxed{\mu_s \geq \frac{v^2}{rg}}$$

- Friction is not always reliable at circular turns if high speed and sharp turns are involved. To avoid dependence on friction, the road are banked at the turn so that the outer part of the road is somewhat lifted up as compared to the inner part.



$$N\cos\theta = mg$$

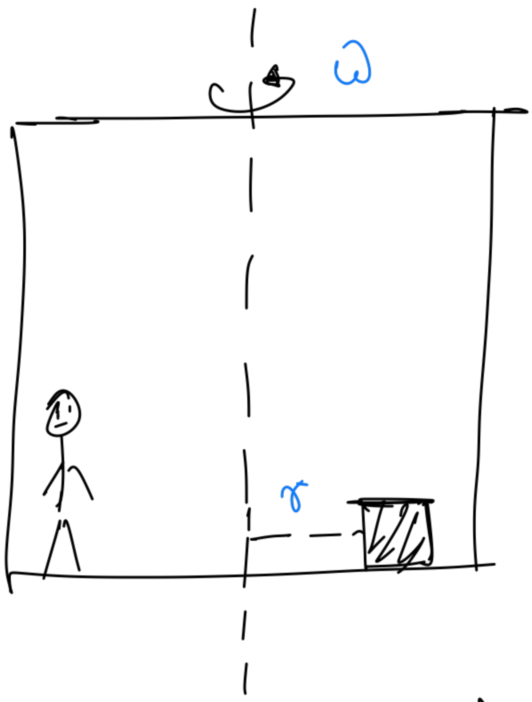
$$N\sin\theta = \frac{mv^2}{r}$$

$$\boxed{\tan\theta = \frac{v^2}{rg}}$$

III Centrifugal force

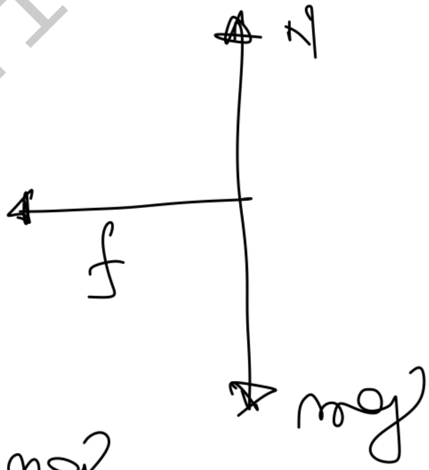
- If the frame translates with respect to an inertial frame with an acceleration $\vec{a_0}$, one must assume the existence of a pseudo force, $-m\vec{a_0}$, acting on a particle of mass m.

- What pseudo force is needed if the frame of reference rotates at a constant angular velocity ω with respect to an inertial frame.



Case 1 : the motion of the box from the ground frame (Inertial frame)

The resultant force on the block,



ω – angular velocity
r – distance from the axis / radius of the circular path

(i) weight, mg
(ii) normal contact force, N
(iii) friction f by the floor.

So, $f = \frac{mv^2}{r} = m\omega^2 r$

– the floor exerts a force of static friction $\boxed{f = m\omega^2 r}$ toward the origin.

- **Case 2**: Now consider the same box when observed from the frame of the rotating cabin.

– The weight and the Normal contact force balance each other but the frictional force $f = m\omega^2 r$ acts on the box towards the origin.

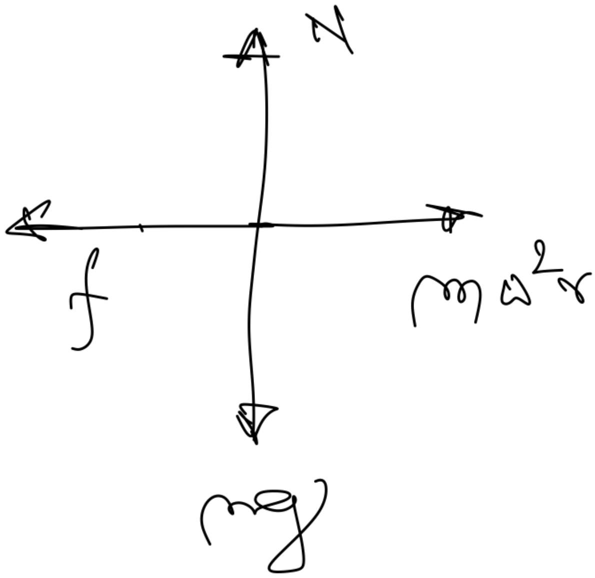
To make the resultant zero, a pseudo force must be assumed which acts on the box away from the centre (radically outward) and has a magnitude $m\omega^2 r$. This pseudo force is called the centrifugal force.

(i) weight, mg

(ii) Normal contact force, N

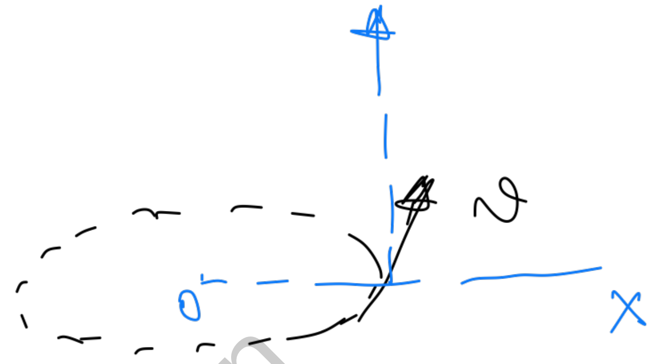
(iii) friction, f

(iv) centrifugal force, $m\omega^2 r$



- Coriolis force: Perpendicular to the velocity of the particle and also perpendicular to the axis of rotation of the frame

- Cause of linear motion/displacement of the body in a uniformly rotating frame.



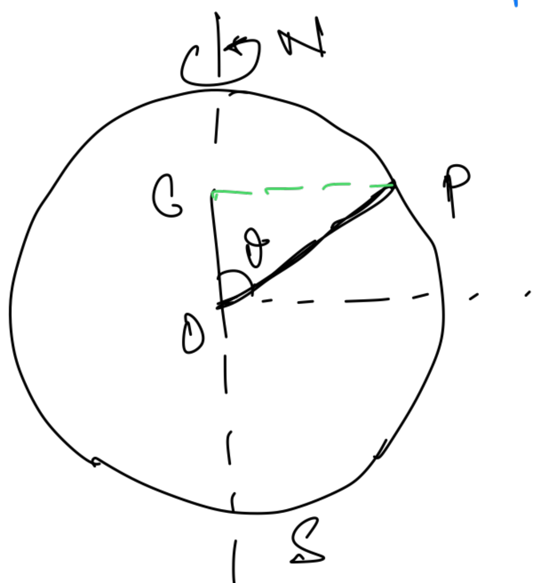
** Centrifugal force acts (or is assumed to act) because we considered the particle from a rotating frame which is non-inertial and still use Newton's law.

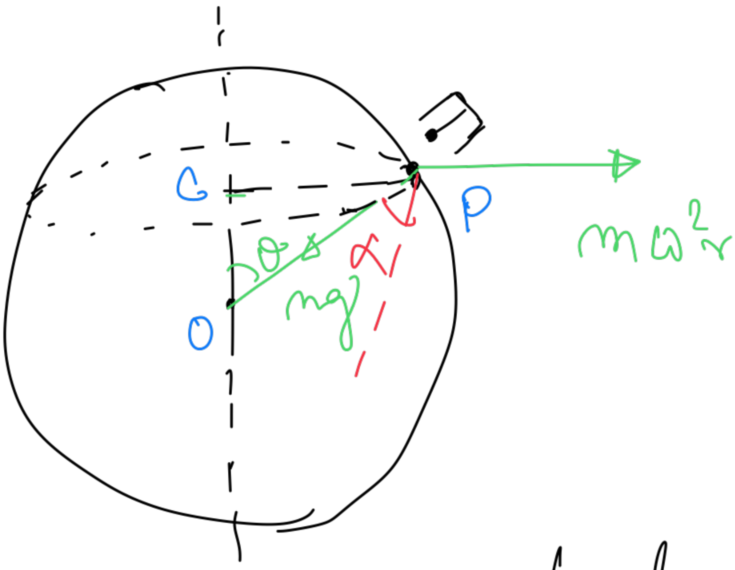
Effects of Earth's rotation on Apparent weight

- $CP = OP \sin\theta$

$= R \sin\theta$

R is the radius of the earth.





- Looking from the earth's frame, the particle of m is in equilibrium and the forces on it are

(i) gravitational attraction mg towards the centre of earth.

(ii) centrifugal force, $m\omega^2 r$ towards CP

(iii) The tension in the string T along the string.

The resultant of mg and $m\omega^2 r$

$$= \sqrt{mg^2 + m^2\omega^4 r^2 + 2(mg)(m\omega^2 r)\cos(90^\circ+\theta)}$$

$$= mg'$$

where $g' = \sqrt{g^2 - \omega^2 R \sin^2\theta\,(2g - \omega^2 R)}$ —— (a)

– The direction of this resultant makes an angle α with the vertical of, where,

$$\tan\alpha = \frac{m\omega^2 R \sin(90^\circ + \theta)}{g - \omega^2 R \sin^2\theta}$$

So, Tension, $T = mg'$

from eq^n (a), It is clear that,

$2g > \omega^2 R$ so, $g' < g$

– So the weight observed (mg') is less than its true weight (mg).

Apparent weight

for $\theta = 90^\circ$, $\sin\theta = 1$.

from eq^n (a) –

$$g' = g - \omega^2 R$$

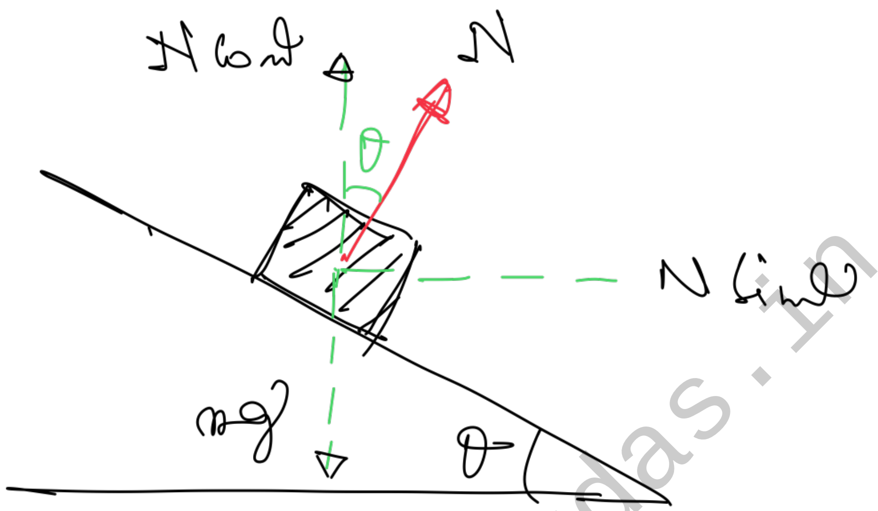
$$\text{or, } mg' = mg - m\omega^2 R$$

and for $\theta = 0^\circ$, $g' = g$

At pole, Apparent weight is same as true weight. $mg' = mg$.

Example:

Circular track of radius 600 m. Average speed 180 km/hr. Find the angle of banking of the track.

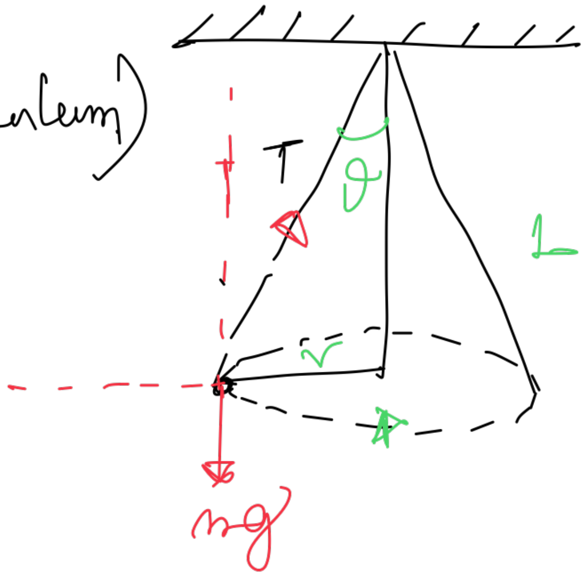


$$N\cos\theta = mg$$

$$N\sin\theta = m\,v^2/r$$

$$\text{or, } \tan\theta = \frac{v^2}{rg}$$

Example: (Conical Pendulum)



$$T\cos\theta = mg$$

$$T\sin\theta = \frac{mv^2}{r}$$