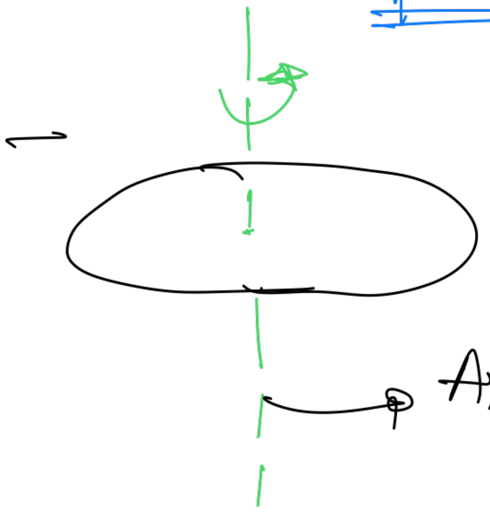
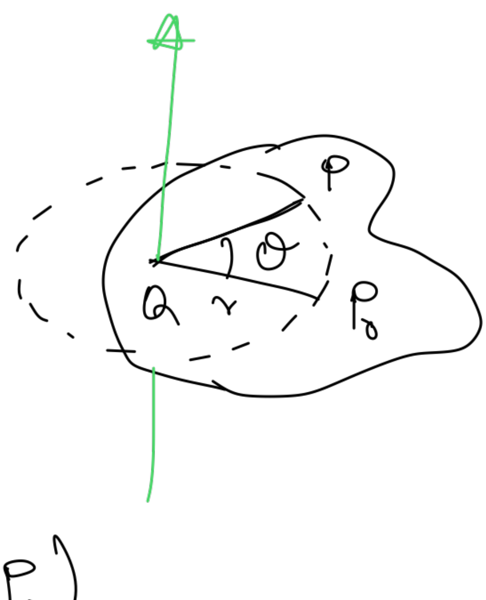
Rotational Mechanics



Axis of Rotation.

- Sometimes the axis may not pass through the body.
- If each particle of a rigid body moves in a circle, with centers of all the circles on a straight line and with planes of the circles perpendicular to this line, we say the body is rotating about this line. The straight line itself is called the axis of rotation.

Kinematics



- Initial position is at P.
- At time t the angular position is θ ($\angle PQP_0$)

- Average angular velocity,

$$\omega = \frac{\Delta\theta}{\Delta t}$$

- Instantaneous angular velocity,

$$\omega = \frac{d\theta}{dt}$$

- S.I unit for angular velocity is radian / sec.

Quite often the angular velocity is given in revolutions per second (rev/s).

1 rev = 2π radian.

- If $\omega = \frac{d\theta}{dt}$ = constant, it is rotating with uniform angular velocity.

$$\theta = \omega t.$$

- If $\omega = \frac{d\theta}{dt} \neq$ constant, acceleration or deceleration comes into the picture.

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

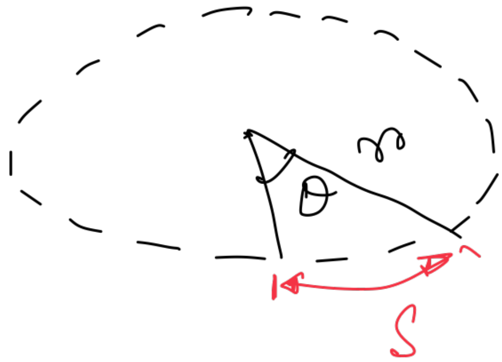
If α is constant,

$$\boxed{\begin{aligned} \omega &= \omega_0 + \alpha t \\ \omega^2 &= \omega_0^2 + 2\alpha\theta \\ \theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \end{aligned}}$$

- Given the axis of rotation, the body can rotate in two directions. It may be clockwise or anticlockwise. One has to define the positive rotation.

❋ Relation between the linear motion of a particle of a Rigid Body and its Rotation.

$$S = r\theta$$



Now,

$$\frac{ds}{dt} = r \cdot \frac{d\theta}{dt}$$

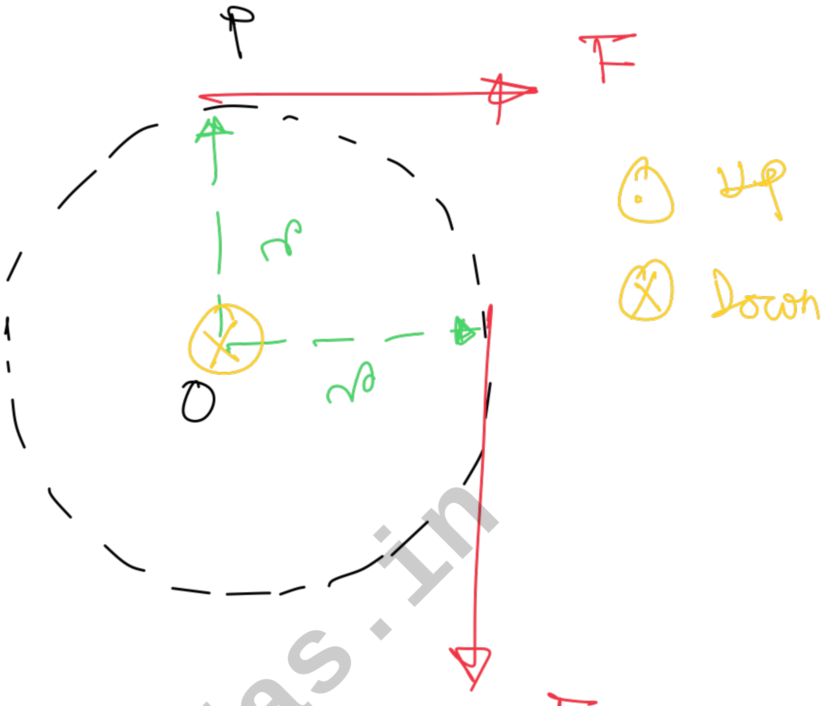
$$v = r\omega$$

And, $\alpha_t = \frac{dv}{dt} = r \cdot \frac{d\omega}{dt} = r\alpha$

$\alpha_t \rightarrow$ Tangential Acceleration.

$\alpha \rightarrow$ Angular Acceleration

Torque of a force about the axis of rotation.



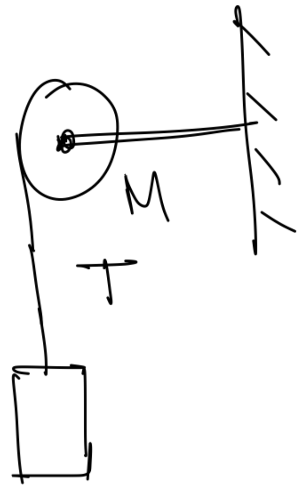
– Consider a force $\vec{F}$ acting on a particle P. The origin is O. Let $\vec{r}$ is the position vector.

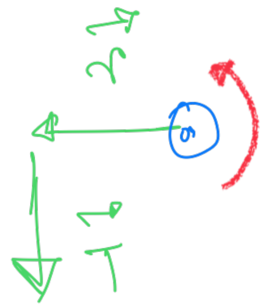
We define the torque of the force F about O as –

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

– is a vector quantity having its direction perpendicular to $\vec{r}$ and $\vec{F}$.

Example





- Mass of the pulley is M.
- Tension in the rope T

(a) Forces acting on the pulley

(i) Mg, vertically downward.

(ii) Tension T along the rope.

(iii) Contact force N by the support at the centre.

So, $N = T + Mg$

Centre of mass of the pulley is at rest.

(b) The torque of the contact force N is zero. Force Mg passes through the centre of mass the axis of rotation.

The tension T is along the tangent of the rim. So, torque,

$$\tau = T \cdot r$$ (positive, and it will try to rotate the pulley anticlockwise).

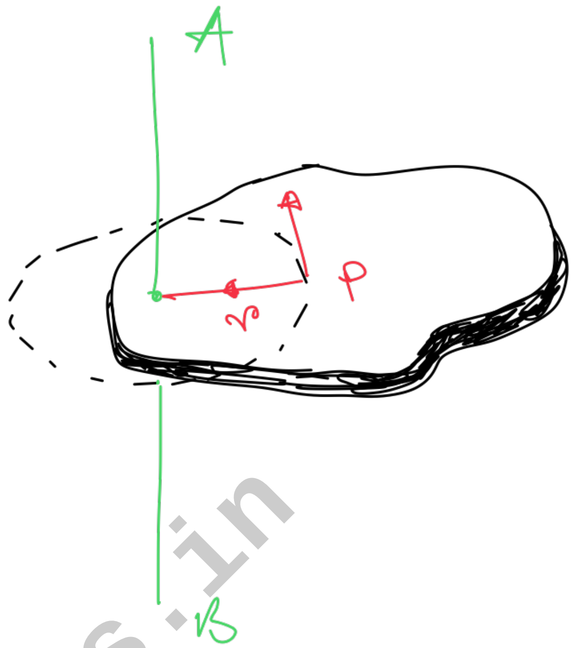
* If there are more than one forces $\vec{F_1}, \vec{F_2}, \ldots$, we have to get separately the torques of the individual forces and then add them to get the total Torque.

$$\vec{\tau} = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + \ldots$$

– Even if the external forces are zero, there can be some angular velocity.

– If the forces act on the same particle add the forces and then take the torque of the resultant.

Moment of Inertia



— Consider a particle P of mass m rotating in a circle of radius r.

The radial acceleration of the particle

$$= \frac{v^2}{r} = \omega^2 r$$

Thus radial force on it $= m\omega^2 r$

The tangential acceleration of the particle $= \frac{dv}{dt}$

Thus the tangential force on it,

$$= m\frac{dv}{dt} = mr\frac{d\omega}{dt} = mr\alpha$$

The torque of $m\omega^2 r$ about AB is zero as it intersects the axis.

Thus the torque of the resultant force acting on P is $mr^2\alpha$.

So, total Torque,

$$\tau_{total} = \sum_i m_i r_i^2 \alpha = I\alpha$$

where, $I = \sum_i m_i r_i^2$

$$\boxed{\tau = I\alpha}$$

I is called moment of Inertia of the body about the axis of rotation.

$m_i \to$ mass of i^{th} particle.

$r_i \to$ perpendicular distance from the axis.

— we have,

$$\tau_{total} = \sum_i (\vec{r}_i \times \vec{F}_i)$$; where F_i is the resultant force acting on i^{th} particle.

$$\tau_{total} = \sum_i \vec{r}_i \times \left(\sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{ext} \right)$$

→ $\tau = I\alpha$; $F = Ma$.

$I = \sum_i m_i r_i^2$ depends on the object of the axis.
changing axis changes r_i and hence I.

* $\tau = I\alpha$ is not an independent rule of nature. It is derived from the more basic Newton's laws of motion.

Bodies in Equilibrium

– The centre of mass of a body remains in equilibrium if the external forces acting on the body is zero. ($F = ma$)

– A body remains in rotational equilibrium if the total external torque acting on the body is zero. ($\tau = Ia$)

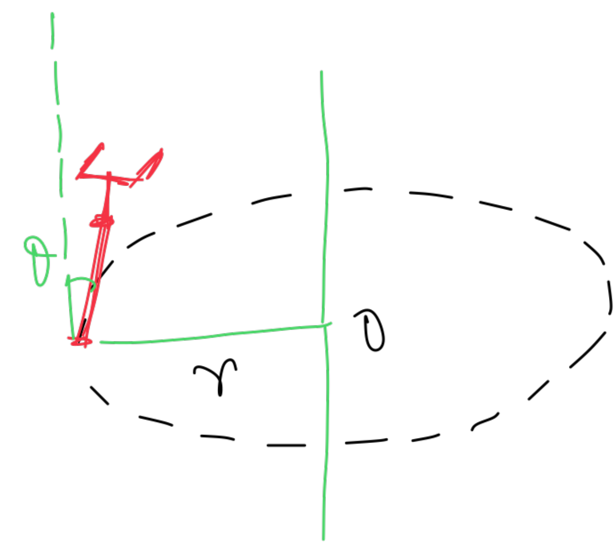
i.e. $\tau = 0$

– The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. The centre of mass goes higher on being slightly displaced.

- It is called unstable if it gets further displaced. The centre of mass goes lower.

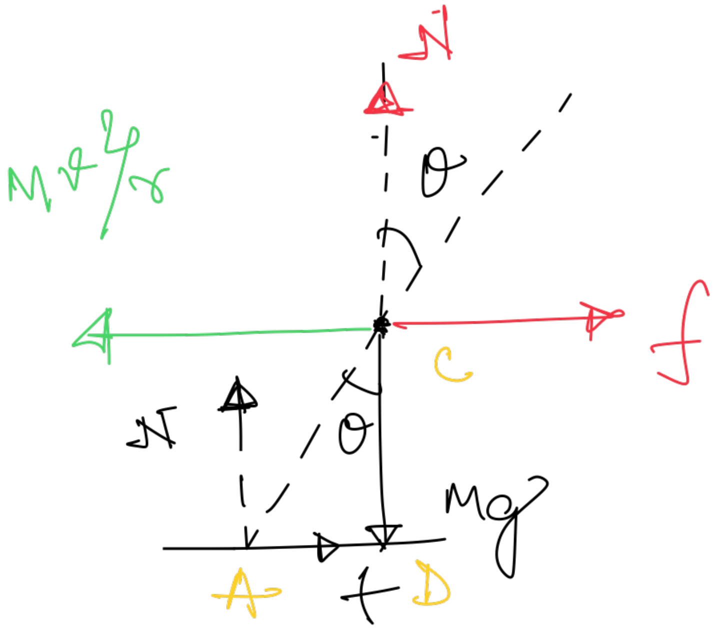
- If it can stay in equilibrium even after being slightly displaced and released, it is said to be in neutral equilibrium. The centre of mass stay at the same.

Bending of a cyclist on a Horizontal Turn.



- Suppose a cyclist is going at a speed v on a circular horizontal road of radius r which is not banked.

The centre of mass C of the system is going in a circle with a centre O and radius r.



– The frame is rotating at an angular speed $\omega = v/r$. In this frame the system is at rest. Since we are working from a rotating frame of reference, we will have to apply centrifugal force on each particle.

The net centrifugal force on the system will be $M\omega^2 r$ or Mv^2/r.

The cycle is bent at an angle θ with the vertical. The forces are,

(i) weight Mg.
(ii) Normal force N
(iii) friction f
(iv) centrifugal Force Mv^2/r.

As the system is at rest, the total external force and the total external torque must be zero.

Consider the point A.

The torques of N and f about A are zero because these forces pass through A.

For rotational equilibrium,

$$Mg(AB) = \frac{Mv^2}{r}(CB)$$

$$\frac{AB}{CB} = \frac{v^2}{rg}$$

or, $$\boxed{\tan\theta = \frac{v^2}{rg}}$$

$\theta = \tan^{-1}\frac{v^2}{rg}$; the angle the cyclist bends with the vertical.

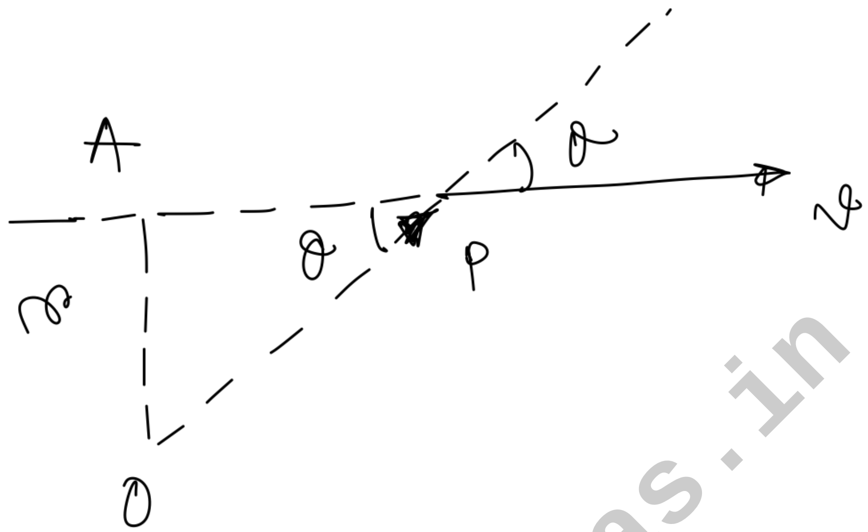
Angular Momentum.

$$\vec{L} = \sum_i \vec{l_i} = \sum_i (\vec{r_i} \times \vec{p_i})$$

$\vec{L}$ = Angular Momentum.

$\vec{r}$ = positional vector

$\vec{p}$ = Linear Momentum.



Suppose a particle P of mass m moves at a velocity $\vec{v}$. Its angular momentum about a point O is,

$$\vec{l} = \vec{OP} \times (m\vec{v})$$

or, $l = mv\, OP \sin\theta = mvr$

$r = OA = OP \sin\theta$ is the perpendicular distance of the line of motion from O.

The component of $\vec{r} \times \vec{p}$ along the line AB is called the angular momentum of the particle "about AB".

Linear velocity, $v = r\omega$.

so, $L = |\vec{r} \times \vec{p}| = mvr = mr^2\omega$.

$$L = \sum m_i r_i^2 \omega = I\omega$$

$$\boxed{L = I\omega}$$

VIII Conservation of Angular Momentum

$$\vec{L} = \sum (\vec{r}_i \times \vec{p}_i)$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum(\vec{r}_i \times \vec{p}_i)$$

$$= \sum_i \left[\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right]$$

$$= \sum_i \left[\vec{v}_i \times m\vec{v}_i + \vec{r}_i \times \vec{F}_i\right]$$

$$= \sum_i (\vec{r}_i \times \vec{F}_i) = \vec{\tau}_{total}$$

For a rigid body,

$$L = I\omega$$

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha$$

or, $$\frac{dL}{dt} = \tau_{ext}$$

If the total external torque on a system is zero, its angular momentum remains constant.

Angular Impulse

$$J = \int_{t_1}^{t_2} \tau \, dt$$

Here, $\tau = \frac{dL}{dt}$

$$J = \int_{t_1}^{t_2} dL = L_2 - L_1$$

The change in angular momentum is equal to the angular impulse of the resultant torque.

Kinetic Energy of a Rigid Body rotating about a given axis.

The kinetic energy

$$= \frac{1}{2} m v^2$$

$$= \frac{1}{2} m (\omega r)^2$$

$$= \frac{1}{2} (m r^2) \omega^2$$

$$= \frac{1}{2} I \omega^2$$

Rotational kinetic energy.

Power delivered and work done by a Torque.

- The torque produces angular acceleration and the kinetic energy increases. The rate of increase of the kinetic energy equals the rate of doing work on it i.e the power delivered by the torque

$$P = \frac{dW}{dt} = \frac{dK}{dt}$$

$$= \frac{d}{dt}\left(\frac{1}{2} I \omega^2\right) = I\omega \frac{d\omega}{dt} = I\omega\alpha$$

$$= \tau \cdot \omega$$

The work done in an infinitesimal angular displacement $d\theta$ is,

$$dW = \tau \cdot \omega\, dt = \tau \cdot d\theta$$

The work done in an finite angular displacement, θ_1 to θ_2 is,

$$\boxed{W = \int_{\theta_1}^{\theta_2} \tau \cdot d\theta}$$

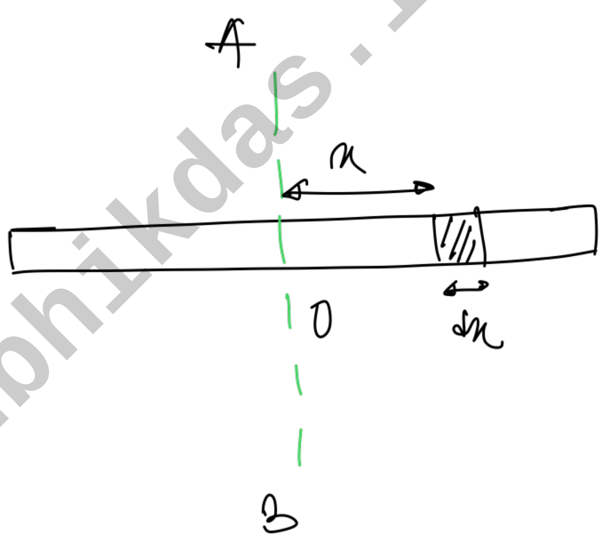
Calculation of Moment of Inertia.

$I = \sum_i m_i r_i^2$; where m_i is the mass of ith particle and r_i is its perpendicular distance from the given line.

— $I = \int r^2 dm$; moment of Inertia of the body about the given line is the sum of the moments of inertia of its constituent elements about the same line.

$\boxed{\text{(A) Uniform rod about a perpendicular bisector}}$

— Consider a uniform rod of mass M and length L. Calculate the moment of Inertia about the bisector AB.



Take the origin at the middle point O of the rod.

Mass per unit length of the rod = M/L.

So, mass of the element $dx = \left(\frac{M}{L}\right)dx$

Distance of dx from AB is x.

Thus, moment of Inertia of the element dx is,

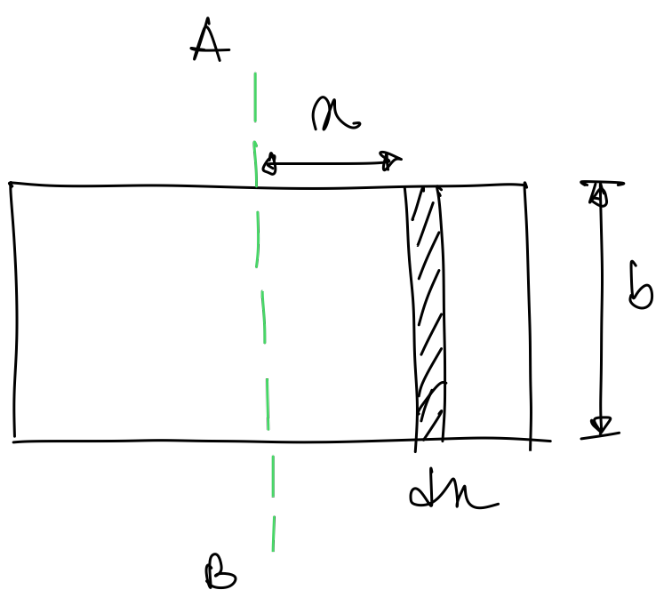
$$dI = \left(\frac{M}{L}\right) dx \cdot x^2$$

So, The moment of Inertia of the entire rod about AB is

$$I = \int_{-L/2}^{L/2} dI = \int_{-L/2}^{L/2} \left(\frac{M}{L}\right) dx \cdot x^2 = \frac{ML^2}{12}$$

(B) Moment of Inertia of a rectangular plate about a line parallel to an edge and passing through the centre.

— Mass per unit area $= \dfrac{M}{bL}$

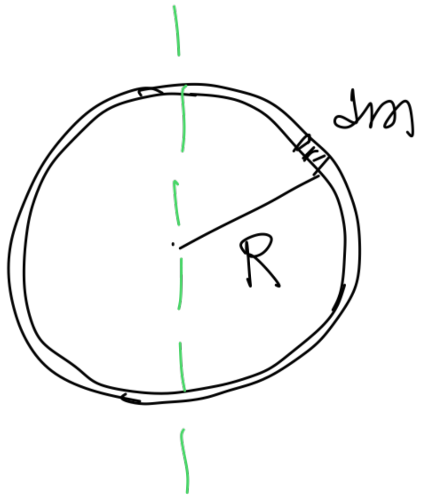


Mass of the strip $= \left(\dfrac{M}{bL}\right) b \cdot dx$

$= \dfrac{M}{L} dx$; $I = \displaystyle\int_{-L/2}^{L/2} \frac{M}{L} x^2 \cdot dx = \frac{ML^2}{12}$

(c) Moment of Inertia of a circular ring about its axis.

$$I = \int r^2 dM$$
$$= r^2 \int dM$$
$$= MR^2$$

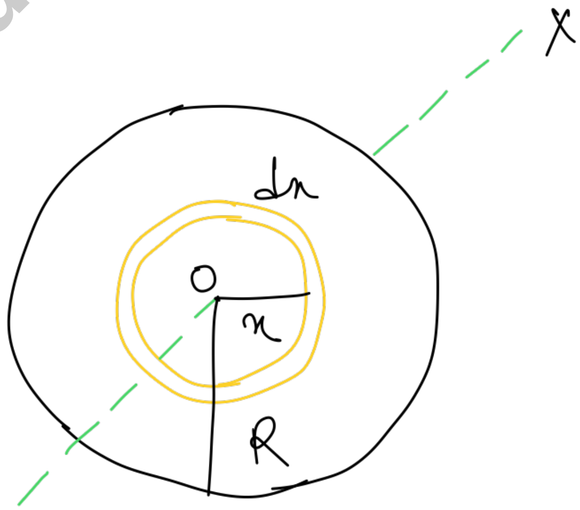


(d) Moment of Inertia of a uniform circular plate about its axis.

- Total Mass M & radius R.

Mass per unit Area

$$= \frac{M}{\pi R^2}$$



Mass of the ring

$$= \frac{M}{\pi R^2} (2\pi x) dx = \frac{2Mx\,dx}{R^2}$$

The moment of Inertia of the ring about OX is,

$$dI = \frac{2Mx\,dx}{R^2} \cdot x^2$$

Thus, the moment of Inertia of the plate about OX is,

$$I = \int_0^R \frac{2Mx\,dx}{R^2} \cdot x^2 = \int_0^R \frac{2M}{R^2} x^3 \cdot dx$$

$$= \frac{MR^2}{2}$$

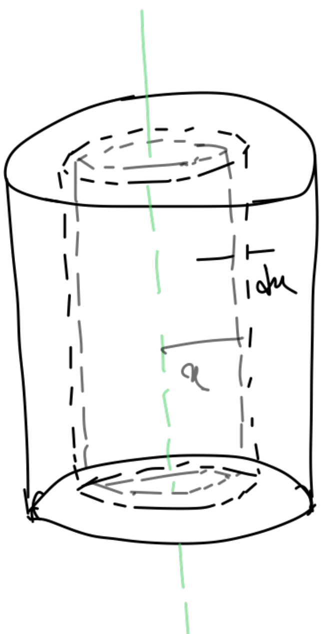
(e) $\boxed{\text{Moment of Inertia of a hollow cylinder.}}$

$$I = \int r^2 dm = R^2 \int dm = MR^2$$

Same as circular ring.

(f) $\boxed{\text{Moment of Inertia of a uniform solid cylinder about its axis}}$

- Mass is M, if radius is R and length is l.



Mass per unit volume,

$$\rho = \frac{M}{\pi R^2 l}$$

Mass of the hollow cylinder –

$$\frac{M}{\pi R^2 l}(2\pi x)\,dx.l = \frac{2M}{R^2}\,x.dx$$

So, $$dI = \left[\frac{2M}{R^2}.x.dx\right].x^2$$

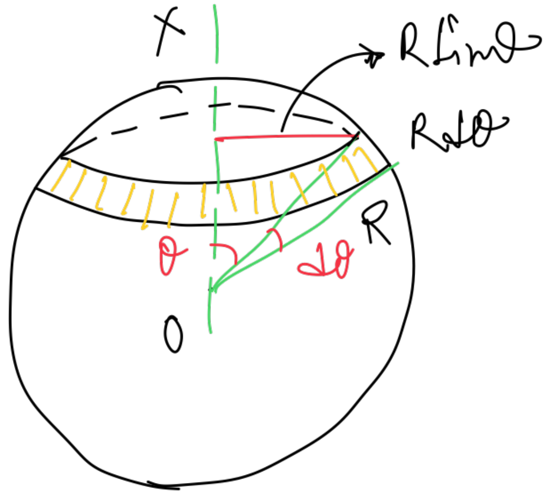
Then, the moment of Inertia of the solid cylinder is,

$$I = \int_0^R \frac{2M}{R^2}\,x^3\,dx = \frac{MR^2}{2}.$$

* The formula does not depend on the length of the cylinder.

(9) Moment of Inertia of a uniform hollow sphere about a diameter

– Mass M, Radius R.



Mass per unit area of the sphere $= \dfrac{M}{4\pi R^2}$

Area of the ring,

$$(2\pi R\sin\theta)\cdot R\,d\theta$$

The Mass of the ring, $\frac{M}{4\pi R^2}\cdot(2\pi R\sin\theta)\cdot R\,d\theta$

$$= \frac{M}{2}\sin\theta\,d\theta$$

So, $dI = \left(\frac{M}{2}\sin\theta\,d\theta\right)\cdot(R\sin\theta)^2$

$$= \frac{MR^2}{2}\sin^3\theta\,d\theta$$

As θ increases from 0 to π, the element rings cover the whole spherical surface.

So, the moment of Inertia of the hollow sphere,

$$I = \int_0^{\pi}\frac{MR^2}{2}\sin^3\theta\,d\theta$$

$$= \frac{MR^2}{2}\int_0^{\pi}(1-\cos^2\theta)\cdot\sin\theta\,d\theta$$

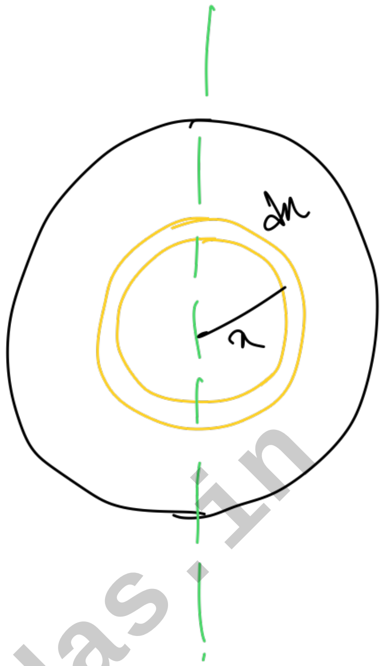
$$= \frac{MR^2}{2}\int_0^{\pi}-(1-\cos^2\theta)\,d(\cos\theta)$$

$$= \frac{2}{3}MR^2$$

④ $\boxed{\text{Moment of Inertia of a uniform solid sphere about a diameter}}$

— Mass M, Radius R.

So, Mass per unit volume $= \dfrac{M}{\frac{4}{3}\pi R^3}$



$$= \frac{3M}{4\pi R^3}$$

Volume of the considered area $= (4\pi x^2)\,dx$

So, Mass $= \dfrac{3M}{4\pi R^3} \cdot (4\pi x^2)\,dx$

$$= \frac{3M}{R^3}\,x^2\,dx$$

The thin spherical shell can be considered as a hollow sphere of radius x.

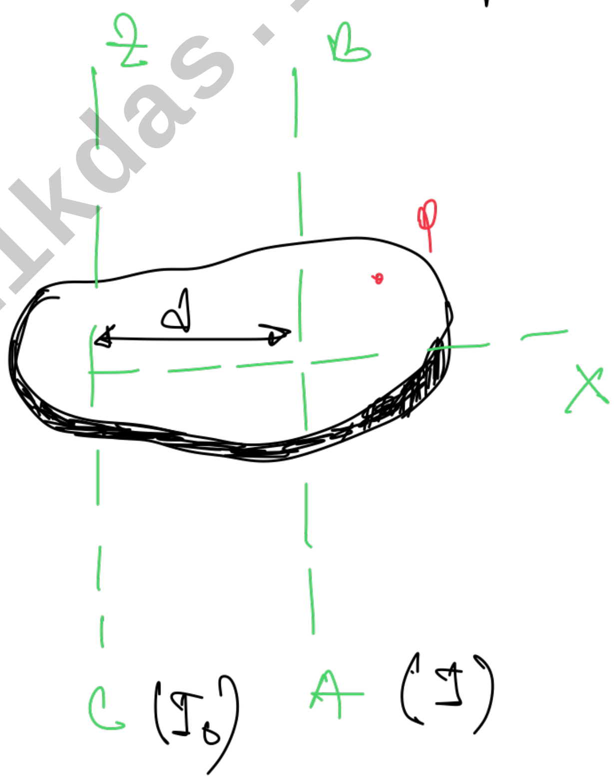
Then, $dI = \dfrac{2}{3}\left(\dfrac{3M}{R^3}\,x^2\,dx\right).\,x^2 = \dfrac{2M}{R^3}\,x^4\,dx$

So, $I = \int_0^R \frac{2M}{R^3} x^4 \, dx$

$= \frac{2}{5} M R^2$

* Theorem of Parallel axes.

Let I and I_0 be the moment of Inertia of a body about AB and CZ respectively.



$I = I_0 + Md^2$

M is mass of the body.
d is the perpendicular distance between AB & CZ.

* Theorem of perpendicular Axes

$I_z = I_x + I_y$

VIII Kinetic Energy of a body in combined Rotation and Translation.

— Consider a body in combined translational and rotational motion in the lab frame.

Suppose in the frame of the centre of mass, the body is making a pure rotation with an angular velocity ω.

The centre of mass itself is moving in the lab frame at a velocity $\vec{v}_0$. The velocity of a particle of mass m_i is $\vec{v}_{i,cm}$ with respect to the centre of mass frame and $\vec{v}_i$ with respect to the mass frame.

We have,

$$\vec{v}_i = \vec{v}_{i,cm} + \vec{v}_0$$

The kinetic energy of the particle in the lab frame is,

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \left(\vec{v}_{i,cm} + \vec{v}_0\right) \cdot \left(\vec{v}_{i,cm} + \vec{v}_0\right)$$

$$= \frac{1}{2} m_i v_{i,cm}^2 + \frac{1}{2} m_i v_0^2 + \frac{1}{2} m_i \left(2 \vec{v}_{i,cm} \cdot \vec{v}_0\right)$$

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i v_{i,cm}^2 + \sum_i \frac{1}{2} m_i v_0^2 + \sum_i (m_i \vec{v}_{i,cm}) \cdot \vec{v}_0$$

Now, $\sum_i \frac{1}{2} m_i v_{i,cm}^2$ is the kinetic energy of the body in the centre of mass frame. In this frame, the body is making pure rotation with an angular velocity ω.

Then, $$\sum_i \frac{1}{2} m_i v_{i,cm}^2 = \frac{1}{2} I_{cm} \omega^2$$

Now, $\dfrac{\sum m_i \vec{v}_{i,cm}}{M}$ is the velocity of the centre of mass in the centre of mass frame which is obviously zero. Then,

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_0^2$$

In case of pure rolling, $v_0 = R\omega$, so that

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

$$= \frac{1}{2} \left(I_{cm} + M R^2 \right) \omega^2 \quad \text{——— (1)}$$

Using the parallel axes theorem,

$I_{cm} + MR^2 = I$, which is the moment of Inertia of the wheel about the line through the point of contact and parallel to the axis.

Thus, $I = \frac{1}{2} I \omega^2$