Work and Energy

– The energy of a moving particle is defined by

$$K(v) = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}$$

and it is called Kinetic Energy.

The kinetic energy of a system of particles is the sum of the kinetic energies of all its constituent particles.

i.e, $K = \sum_i \frac{1}{2}m_i v_i^2$

– A force is necessary to change the kinetic energy of a particle.

– If the resultant force acting on a particle is perpendicular to its velocity, the speed of the particle does not change and hence the kinetic energy does not change.

– Kinetic energy changes only when the resultant force has a tangential component.

- So, the rate of change of kinetic energy,

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv.\frac{dv}{dt}$$

$$= m\frac{dv}{dt}.v$$

$$= F_t v.$$

where F_t is the resultant tangential force.

If the resultant force F makes an angle θ with the velocity,

$$F_t = F\cos\theta$$

$$\frac{dK}{dt} = Fv\cos\theta = \vec{F}.\vec{v} = \vec{F}.\frac{d\vec{r}}{dt}$$

or, $dK = \vec{F}.d\vec{r}$

Work and work-energy Theorem

- The quantity $\vec{F}.d\vec{r} = F dr\cos\theta$ is called the work done by the force $\vec{F}$ on the particle during the small displacement $d\vec{r}$.

So, work done on the particle during a finite displacement –

$$W = \int \vec{F}\cdot d\vec{r} = \int F\cos\theta \, dr$$

$$= \int dK = K_2 - K_1$$

– Work done by the resultant force is equal to the change in its kinetic energy. This is called Work-energy theorem.

If, $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots$

$$W = \int \vec{F}\cdot d\vec{r} = \int \vec{F_1}\cdot d\vec{r} + \int \vec{F_2}\cdot d\vec{r} + \ldots$$

Thus, work done by the resultant force is equal to the sum of the work done by the individual forces.

– The rate of doing work is called the power delivered.

$$\text{Power, } P = \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$$

$\boxed{\text{SI Unit – Joule/Second or 'Watt'.}}$

1 Horsepower = 746 Watt.

VIII Calculation of Work Done

$$W = \int \vec{F} \cdot d\vec{r}$$

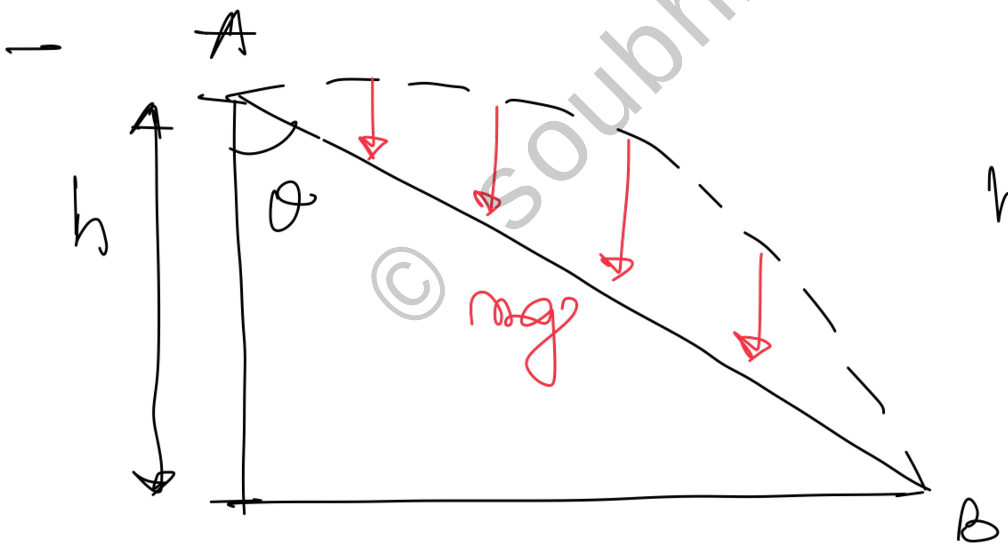
* $\boxed{\text{Constant force}}$

If θ be the angle between the constant force $\vec{F}$ and displacement $\vec{r}$, the work done is

$$W = Fr\cos\theta$$

if $\theta = 0°$, $\cos\theta = 1$; $W = Fr$

if $\theta = 90°$, $\cos\theta = 0$; $W = 0$



$h = AB\cos\theta$

A particle moves from A to B along some curve and $\vec{AB}$ makes an angle θ with the vertical.

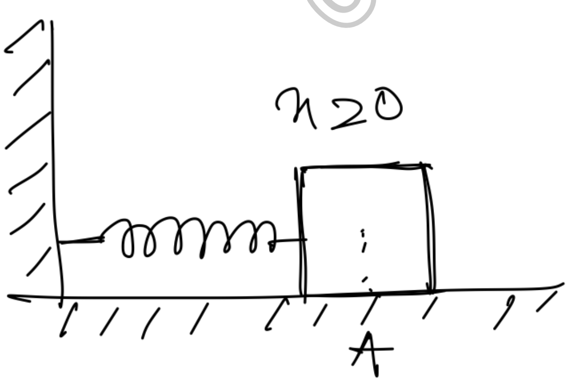
work done by the force of gravity during the transit from A to B is

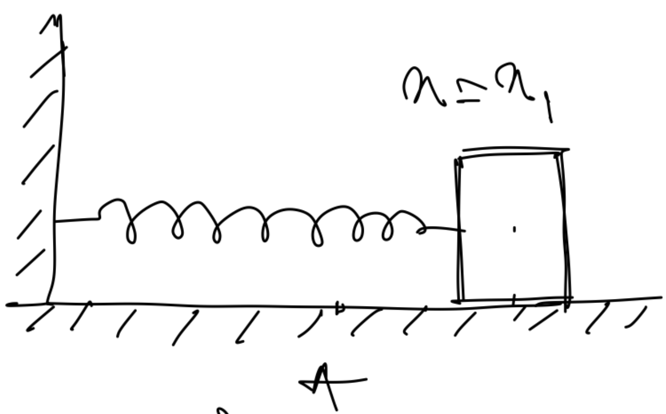
$$W = mg(AB)\cos\theta = mgh$$

– The work done by a constant force is going from A to B depends only on the position of A and B and not on the actual path follow.

If a particle starts from A and reaches to the same point A after some time, the work done by gravity during this round trip is zero.

* $\boxed{\text{Spring force}}$





– The force on the block is K times the elongation of the spring. But the elongation changes as the block moves and so does the force.

- We cannot take $\vec{F}$ out of the integration $\int \vec{F} \cdot \vec{dr}$. We have to write the work done during a small interval in which block moves from x to $x + dx$.

So, $\vec{F} \cdot \vec{dr} = -F\,dx = -kx\,dx$

$$W = \int_0^{x_1} -kx\,dx = \left[-\frac{1}{2}kx^2\right]_0^{x_1} = -\frac{1}{2}kx_1^2$$

or, $$W = \int_{x_1}^{x_2} -kx\,dx = \boxed{\frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2}$$

- The net work done by the spring-force in a round trip is zero.

* **Force perpendicular to velocity**

if $\vec{F} \perp \vec{v}$, $\vec{F} \cdot \vec{dr} = \vec{F} \cdot \vec{v}\,dt$ is zero.

- The change in the kinetic energy of a system is equal to the work done on the system by the external as well as the internal forces.

Conservative and Non-conservative Forces

- If the work done by a force during a round trip of a system is always zero, i.e. the work done by a force depends only on the initial and final states and not on the path follow, it is called a conservative force. Otherwise, it is called non-conservative.

- Conservative force → gravity, Coulomb force, force of spring.

- Non-conservative → the force of friction

Potential Energy

- We define the change in potential energy of a system corresponding to a conservative internal force as -

$$U_f - U_i = -W = -\int_i^f \vec{F} \cdot \vec{dr}$$

where, W is the work done by the internal force on the system as the system passes from the initial configuration i to the final configuration f.

- We don't / can't define potential energy corresponding to a nonconservative internal force.

$$U_f - U_i = -W = -(K_f - K_i)$$

or, $U_f + K_f = U_i + K_i$

- The sum of kinetic energy and potential energy is called the total mechanical energy.

- The total mechanical energy of a system remains constant if the internal forces are conservative and the external forces do no work. This is called the principle of conservation of energy.

- According to the work-energy theorem, the work done by all the forces equals the change in the kinetic energy, then,

$$W_c + W_{nc} + W_{ext} = K_f - K_i \quad \text{——(a)}$$

where, $W_c \rightarrow$ work done by conservative force

$W_{nc} \rightarrow$ Work done by non Conservative force

$W_{ext} \rightarrow$ Work done by external forces.

Now, $W_c = -(U_f - U_i)$

from eq^n ②,

$$W_{nc} + W_{ext} = (K_f + U_f) - (K_i + U_i)$$

$$\boxed{W_{nc} + W_{ext} = E_f - E_i}$$

$E = K + U$, total mechanical energy.

- If some of the internal forces are non-conservative, the mechanical energy of the system is not constant.

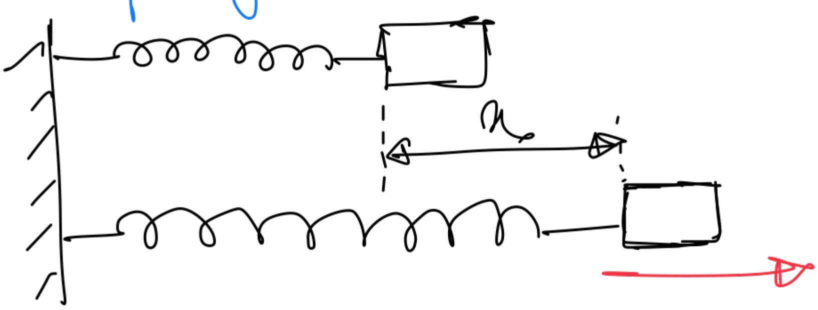
- If the internal forces are conservative, the work done by the external forces is equal to the change in mechanical energy.

- The internal forces acting between the particles of a rigid body do no work in its motion. and we need not consider the potential energy corresponding to these forces.

– The potential energy of a system changes only when the separations between the parts of the system change. The potential energy depends only on the separation between the interacting particles.

– If a block of mass m is raised to a height h above the earth's surface ($h <<$ radius of earth), the potential energy of the 'earth + block' system increases by mgh. If the block descends by a height h, the potential energy decreases by mgh.

III Potential Energy of an Compressed or Extended Spring:



Elastic potential Energy / Strain energy

$$= \frac{1}{2}Kx^2 \; ; \quad K = \text{spring constant}$$

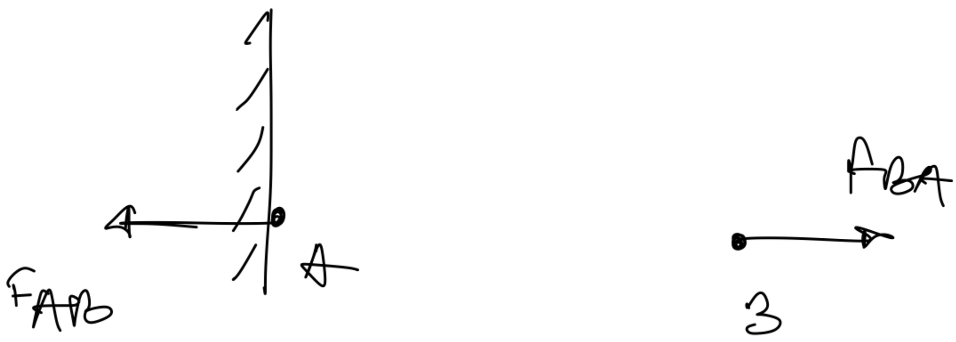
x = Elongation or compression

- The work is positive as the force is towards the right and the particles of the spring, on which the force is acting, also move towards right. Thus the total external work done on the spring is $\frac{1}{2}Kx^2$.

* Energy can never be created or destroyed, it can only be changed from one form into another.

* $E = mc^2$; $c = 3 \times 10^8 \, m/s$

Examples



A & B are charged particles. A is clamped to a fixed point. A & B repel each other by k/r^2.

B has a mass of m and is released from rest with an initial separation r_0 from A.

(i) Find the change in potential energy of the two particle system as the separation increases to a large value.

(ii) What will be the speed of B.

— F_{AB} acting on A & F_{BA} acting on B.

The force F_{AB} does no work, because it acts on A which does not move.

The work done F_{BA},

$$W = \int \vec{F} \cdot \vec{dr} = \int_{r_0}^{\infty} \frac{K}{r^2} dr = \frac{K}{r_0}$$

The change in potential energy of the system is,

$$U_f - U_i = -W = -\frac{K}{r_0}$$

The Total mechanical Energy is conserved,

$$U_f + K_f = U_i + K_i$$

or, $K_f = (U_i + K_i) - U_f$

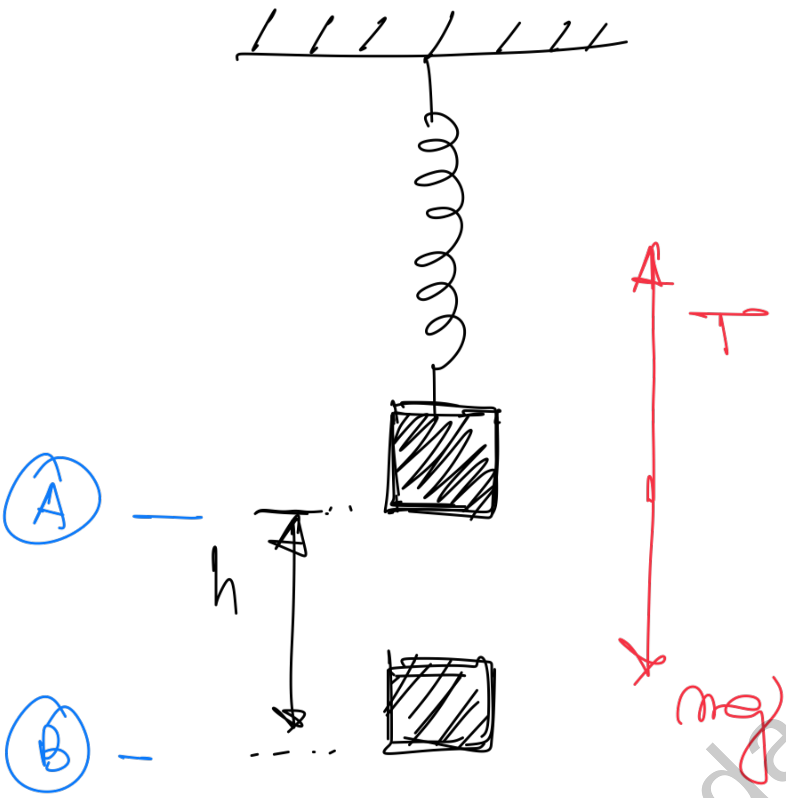
$$\frac{1}{2} m v^2 = K_i - (U_f - U_i)$$

$$= \frac{K}{r_0}$$

or, $v^2 = \frac{2K}{m r_0}$

$$v = \sqrt{\frac{2K}{m r_0}}$$

Example



How far below the equilibrium position, the block comes to an instantaneous rest?

$T = Kx$; x is elongation

for Equilibrium $mg = Kx$

or, $x = mg/K$

At position A, no gravitational potential energy.

The total mechanical energy of the system,

$$\frac{1}{2}Kx^2 + \frac{1}{2}mv^2$$

$$= \frac{m^2g^2}{2K} + \frac{1}{2}mv^2$$

At position B, instantaneous rest:

The elastic potential energy, $\frac{1}{2}K(x+h)^2$

Gravitational potential energy $-mgh$.

Kinetic energy is zero.

As ~~the~~ mechanical energy is conserved,

$$\frac{1}{2}mv^2 + \frac{m^2g^2}{2K} = \frac{1}{2}K(x+h)^2 - mgh$$

$$= \frac{1}{2}K\left(\frac{mg}{K}+h\right)^2 - mgh$$

$$= \frac{1}{2}\frac{(mg+Kh)^2}{K} - mgh$$

$$= \frac{1}{2}\cdot\frac{m^2g^2}{K} + \frac{1}{2}\frac{K^2h^2}{K} + \frac{mg\cdot Kh}{K} - mgh$$

or, $$\frac{1}{2}mv^2 + \cancel{\frac{m^2g^2}{2K}} = \cancel{\frac{m^2g^2}{2K}} + \frac{1}{2}Kh^2 + \cancel{mgh} - \cancel{mgh}$$

or, $mv^2 = kr^2$

or, $r^2 = v^2 \cdot m/k$

$$\boxed{r = v\sqrt{m/k}}$$